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Slide of the Seminar

The Energy Cascade in Compressible Turbulence

Prof. Aluie Hussein

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(P.I. Prof. Luca Biferale)

Università degli Studi di Roma Tor Vergata
C.F. n. 80213750583 – Partita IVA n. 02133971008 - Via della Ricerca Scientifica, 1 – 00133 ROMA

The Energy Cascade in Compressible Turbulence

Hussein Aluie

Department of Mechanical Engineering
and
Laboratory for Laser Energetics

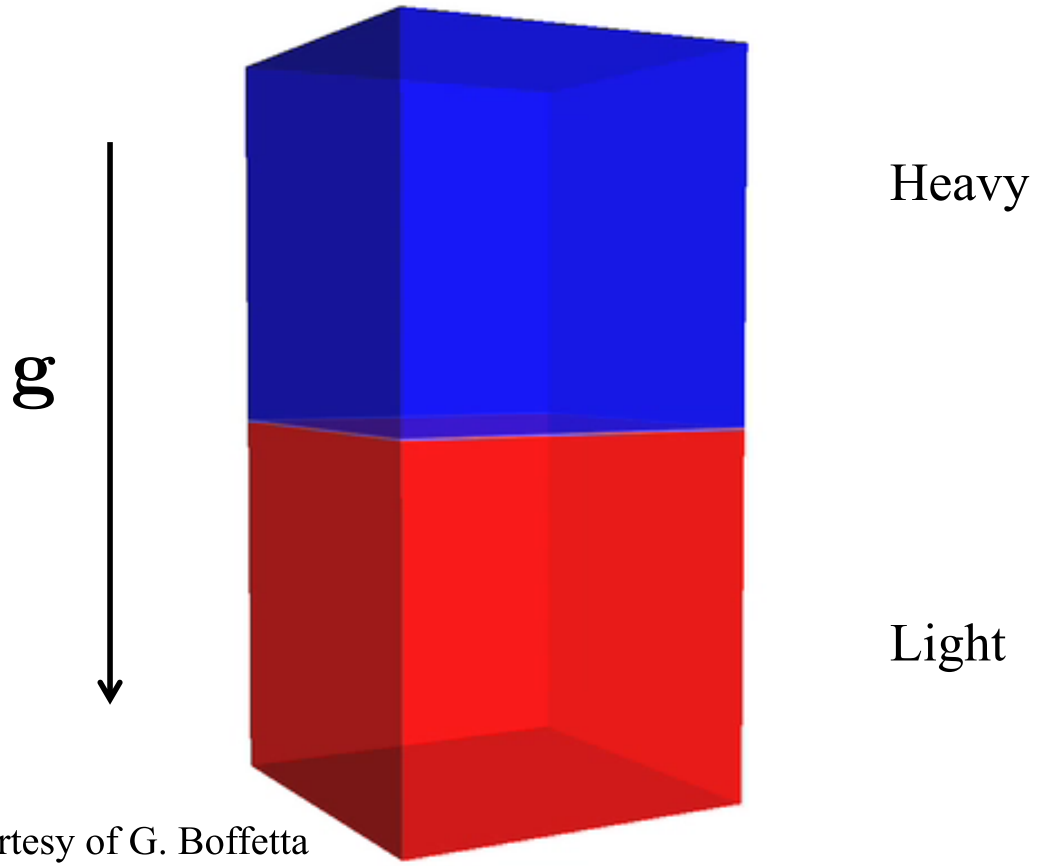
University of Rochester

in collaboration with

Shengtai Li and **Hui Li**
Los Alamos National Lab

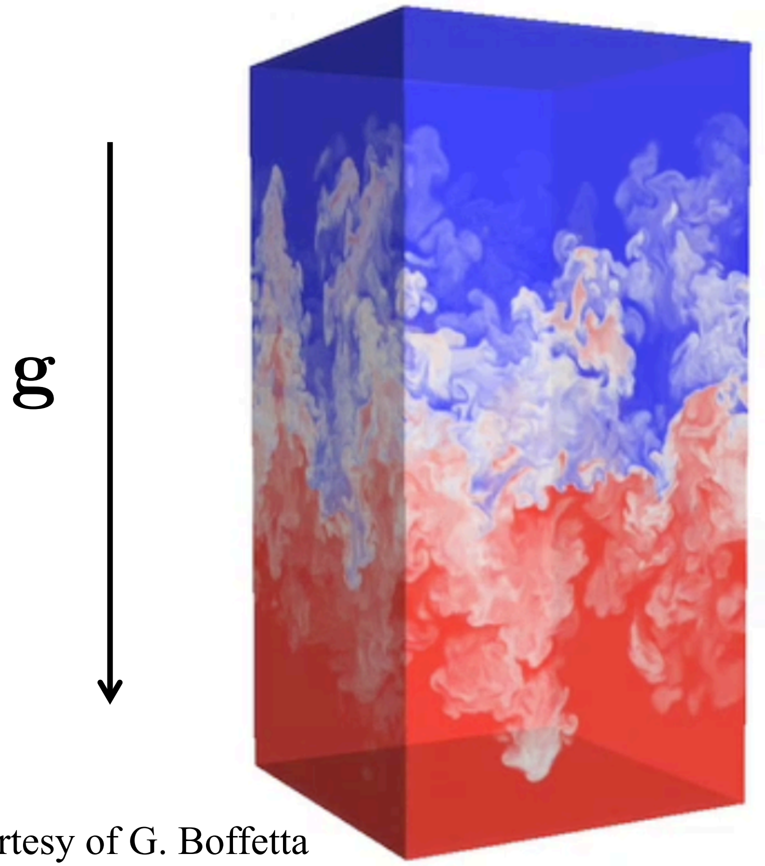
U.S. DEPARTMENT OF ENERGY | Office of Science

Rayleigh-Taylor Instability



movie courtesy of G. Boffetta

Rayleigh-Taylor Instability



movie courtesy of G. Boffetta

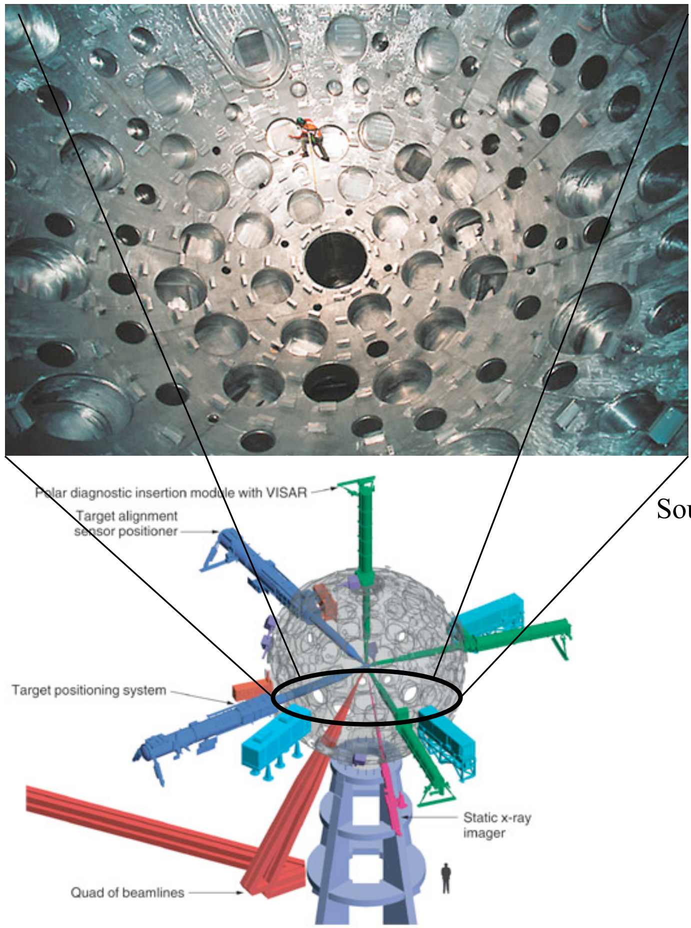
Understand and quantify scale-coupling

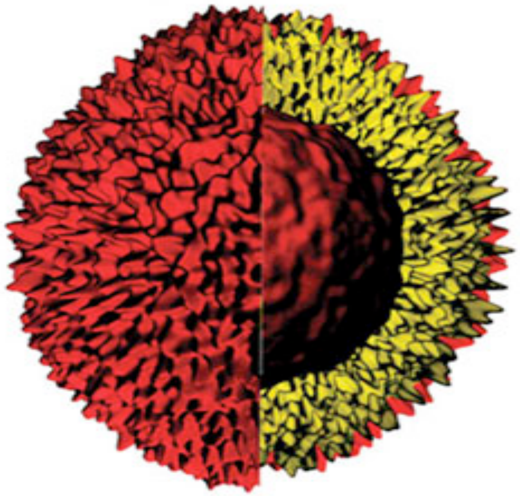
- Modeling small scales
- Coupled multi-scale simulations

Inertial Confinement Nuclear Fusion

National Ignition Facility (Livermore)

$3.5 billion to build
$300 million/yr to operate





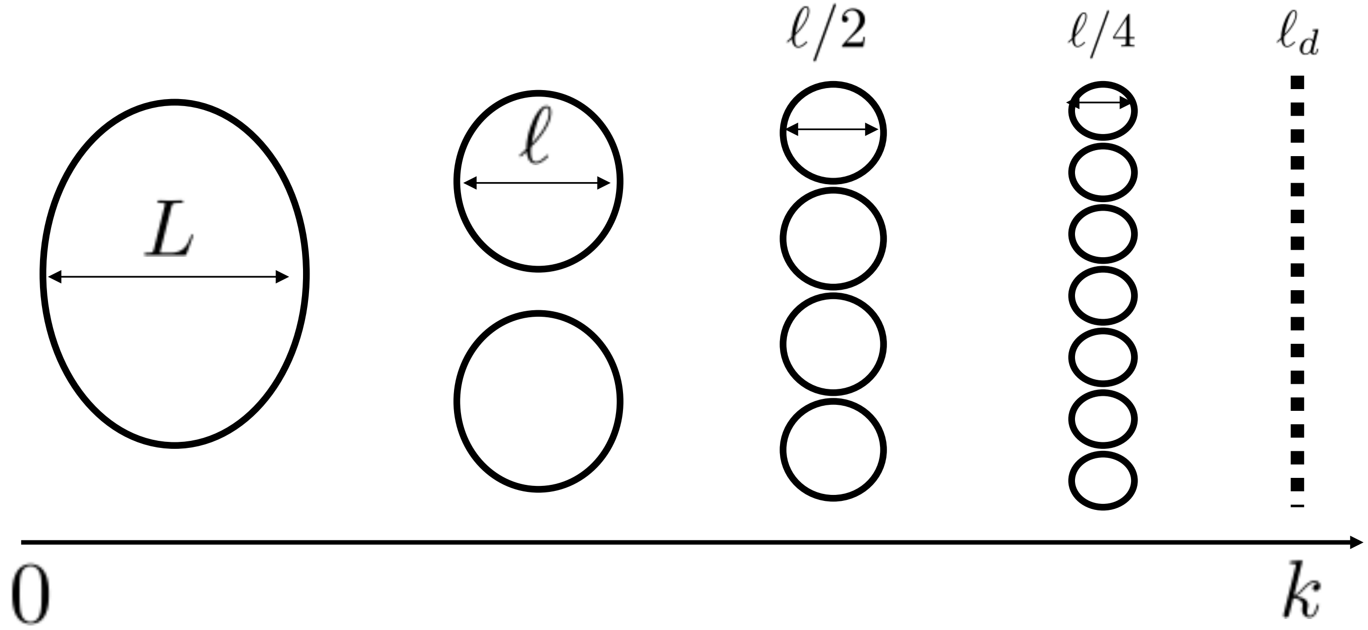
Source: www.llnl.gov

Rayleigh-Taylor Instability is the major obstacle in realizing “Ignition”

Multi-scale interactions & Turbulence

Hierarchy of scales: $L \gg \ell \gg \ell_d$

Fourier wavenumber: $k = 1/\ell$

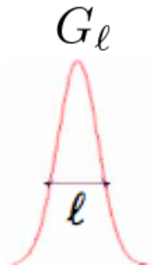


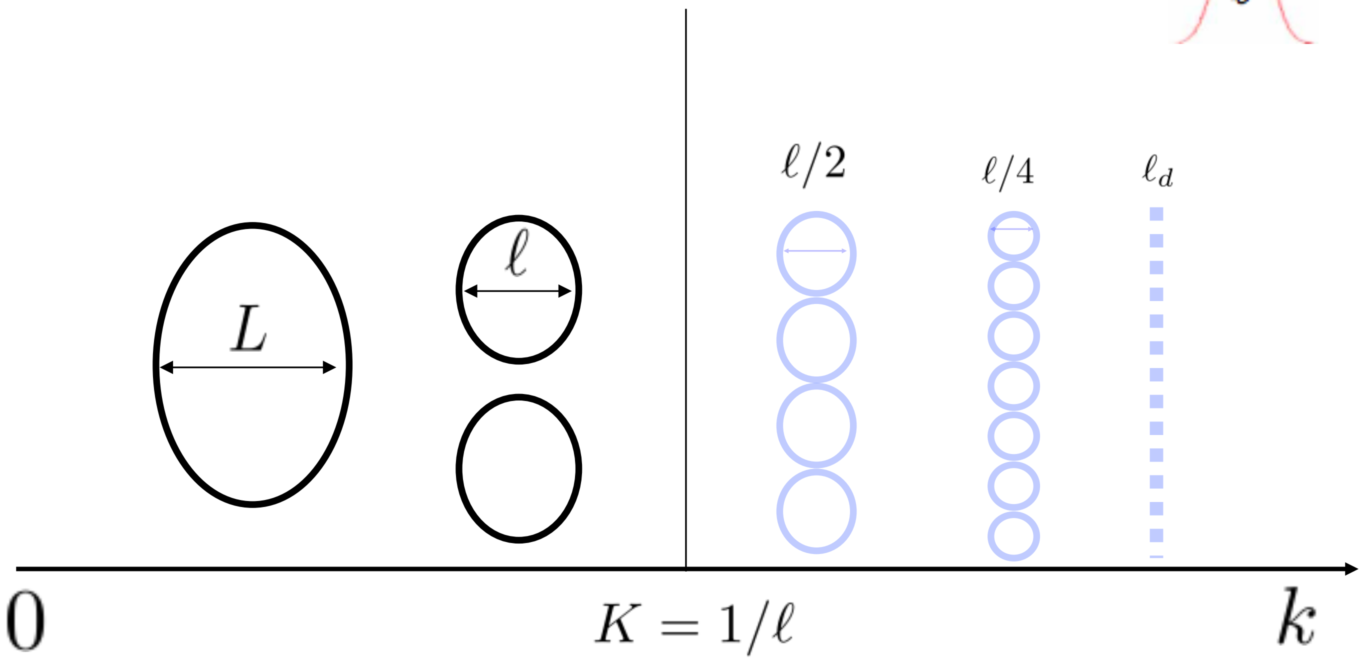
Our Approach

Coarse-graining (Filtering)

[Leonard (1974), Germano (1992),
Eyink (1994),
Ecke, Chen, Eyink et al. (2003)]

$$\overline{\mathbf{u}}_\ell(\mathbf{x}) = \int d\mathbf{r}\; G_\ell(\mathbf{r})\mathbf{u}(\mathbf{x}+\mathbf{r})$$



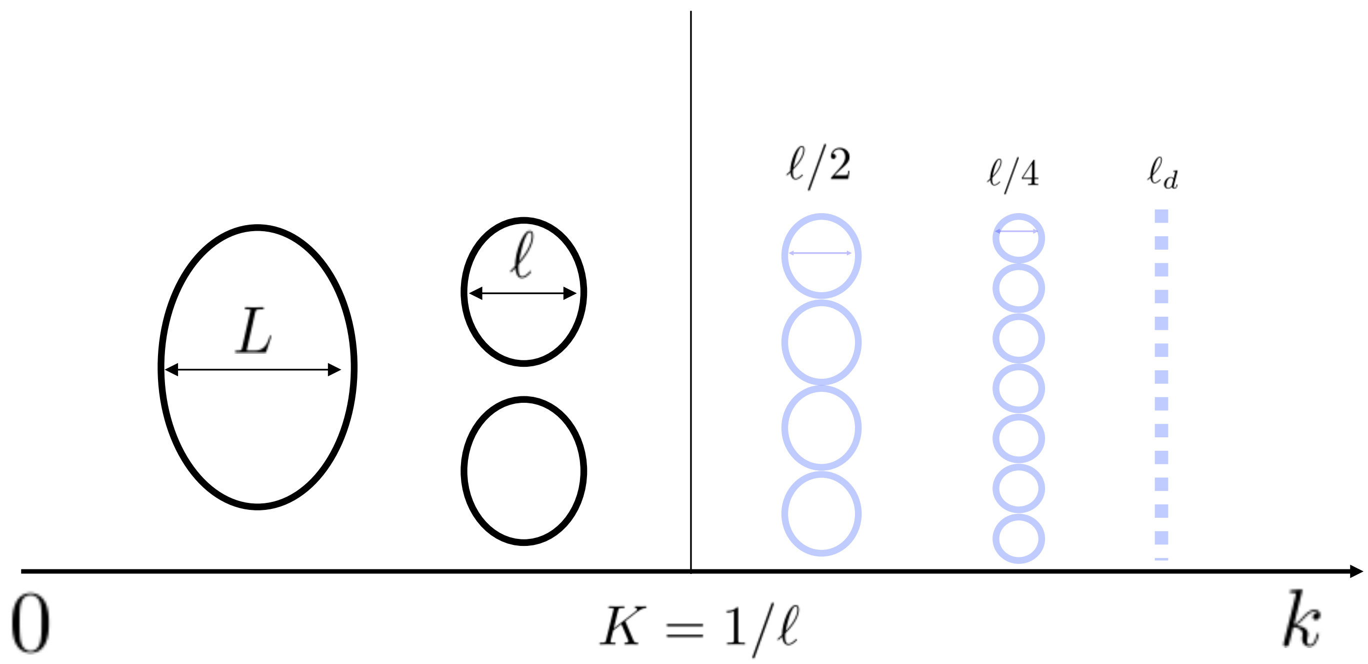


Our Approach

Coarse-grained dynamics

$$\partial_t \overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \boldsymbol{\nabla})\overline{\mathbf{u}} = -\boldsymbol{\nabla}\overline{p} - \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell + \nu \nabla^2 \overline{\mathbf{u}}$$

$$\boldsymbol{\nabla} \cdot \overline{\mathbf{u}} = 0$$



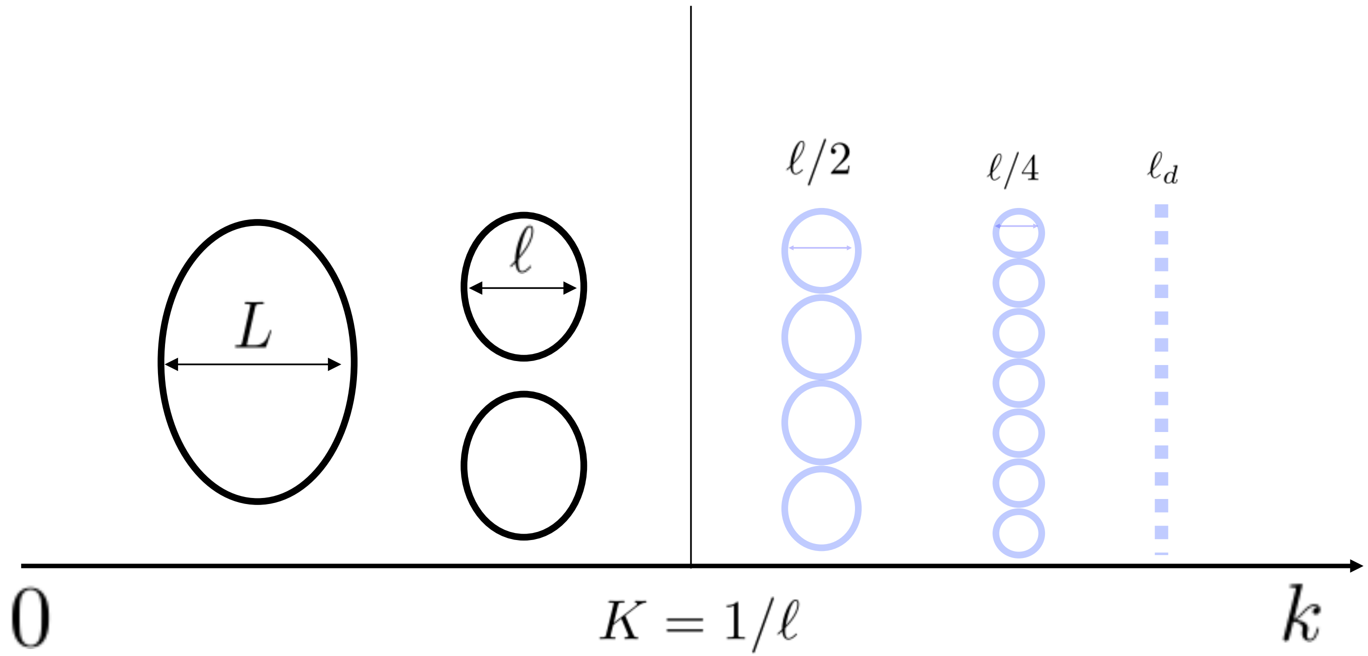
Our Approach

Coarse-grained dynamics

$$\partial_t \overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \boldsymbol{\nabla})\overline{\mathbf{u}} = -\boldsymbol{\nabla}\overline{p} - \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell + \nu \nabla^2 \overline{\mathbf{u}}$$

$$\boldsymbol{\nabla} \cdot \overline{\mathbf{u}} = 0$$

Sub-scale stress: $\boldsymbol{\tau}_\ell = \overline{\mathbf{u}\mathbf{u}}_\ell - \overline{\mathbf{u}}_\ell \overline{\mathbf{u}}_\ell$



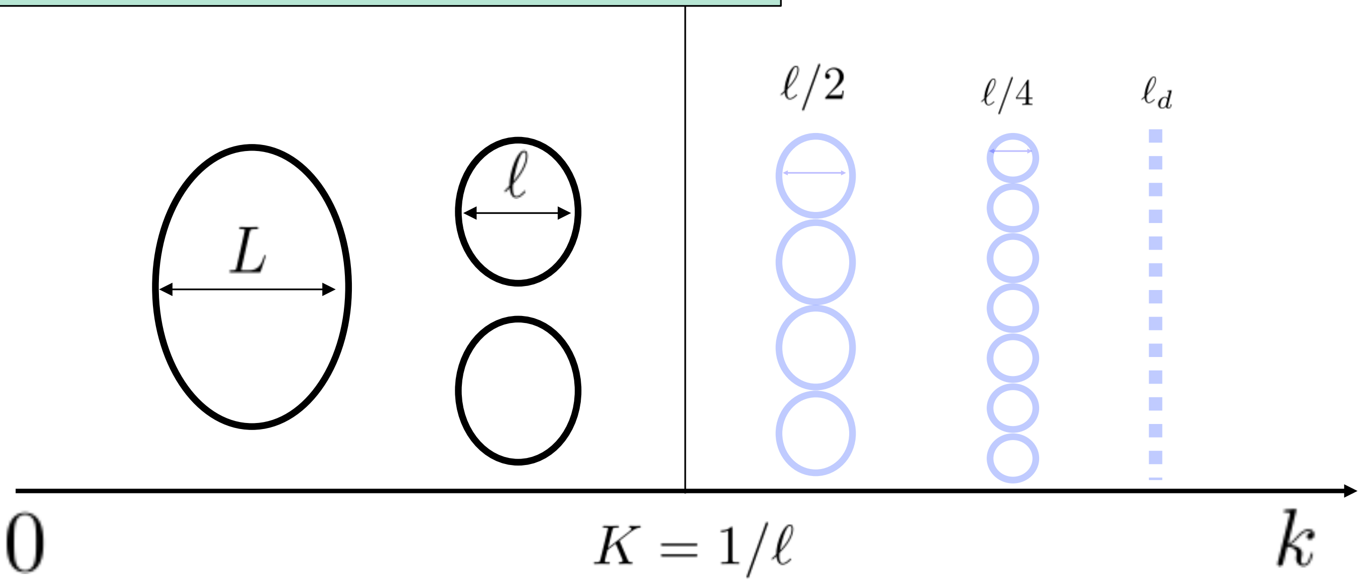
Our Approach

Coarse-grained dynamics

$$\partial_t \overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \boldsymbol{\nabla})\overline{\mathbf{u}} = -\boldsymbol{\nabla}\overline{p} - \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell + \nu \nabla^2 \overline{\mathbf{u}}$$

$$\boldsymbol{\nabla} \cdot \overline{\mathbf{u}} = 0$$

- Every point $\mathbf{x}$ and every instant t
- Variable scale ℓ

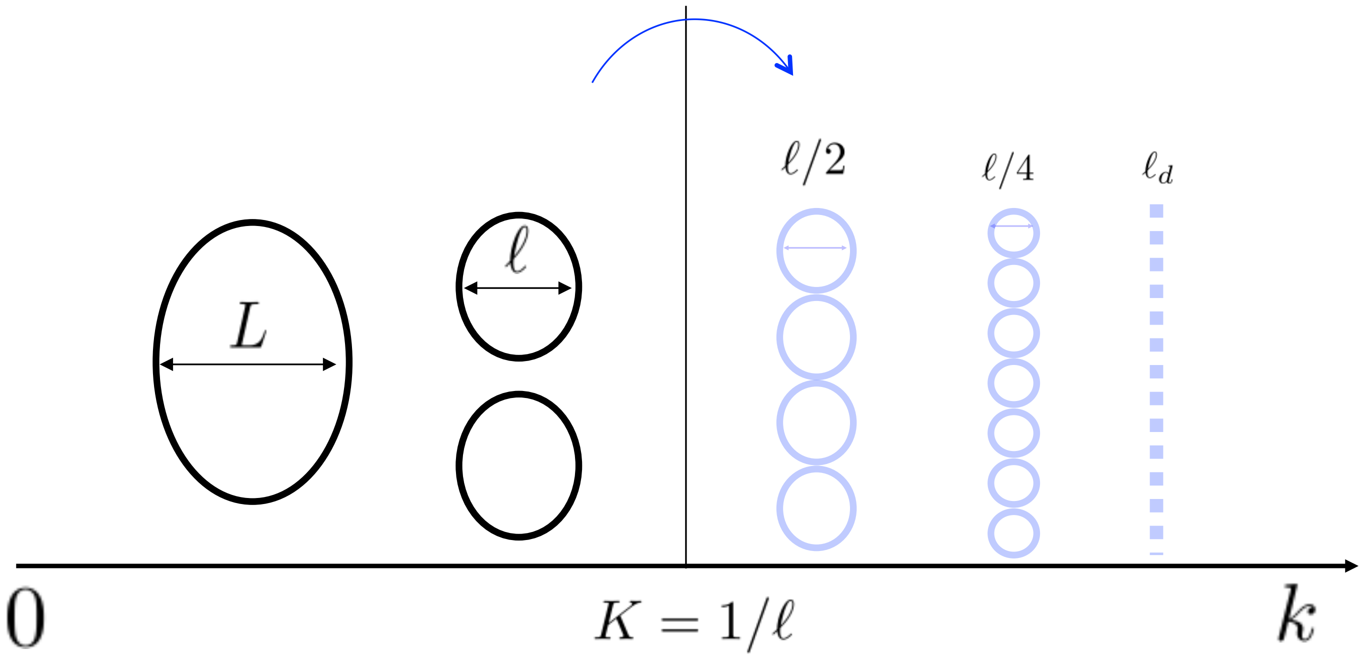


Cascade of Energy

Large-scale energy budget

$$\partial_t \frac{|\overline{\mathbf{u}}|^2}{2} + \boldsymbol{\nabla} \cdot [\cdots] = -\Pi_\ell^E - \nu |\boldsymbol{\nabla} \overline{\mathbf{u}}|^2$$

Energy flux



Cascade of Energy

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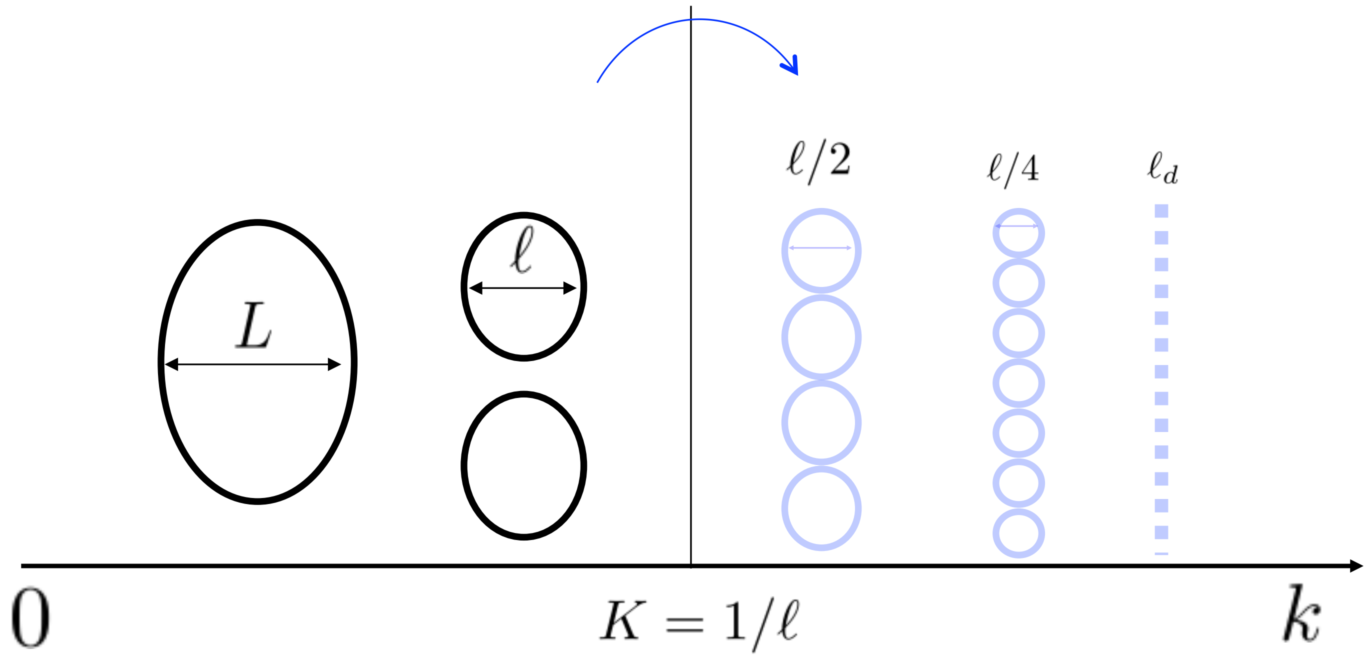
$$\partial_t \frac{|\overline{\mathbf{u}}|^2}{2} + \boldsymbol{\nabla} \cdot [\cdots] = -\Pi_\ell^E - \nu |\boldsymbol{\nabla} \overline{\mathbf{u}}|^2$$

$$\Pi_\ell^E = -\boldsymbol{\nabla} \overline{\mathbf{u}}_\ell : \boldsymbol{\tau}_\ell$$

Energy flux



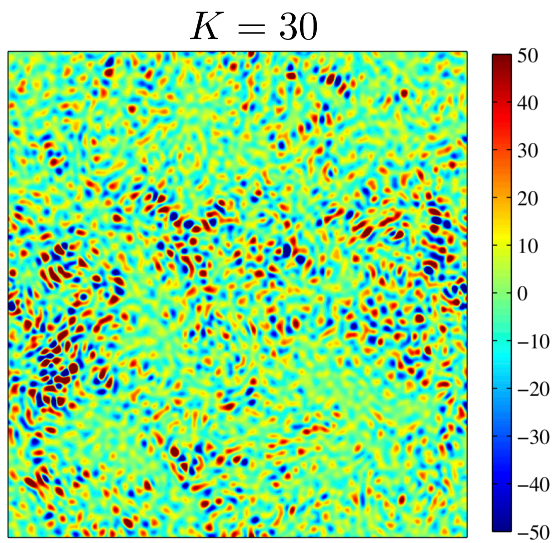
Cascade of Energy

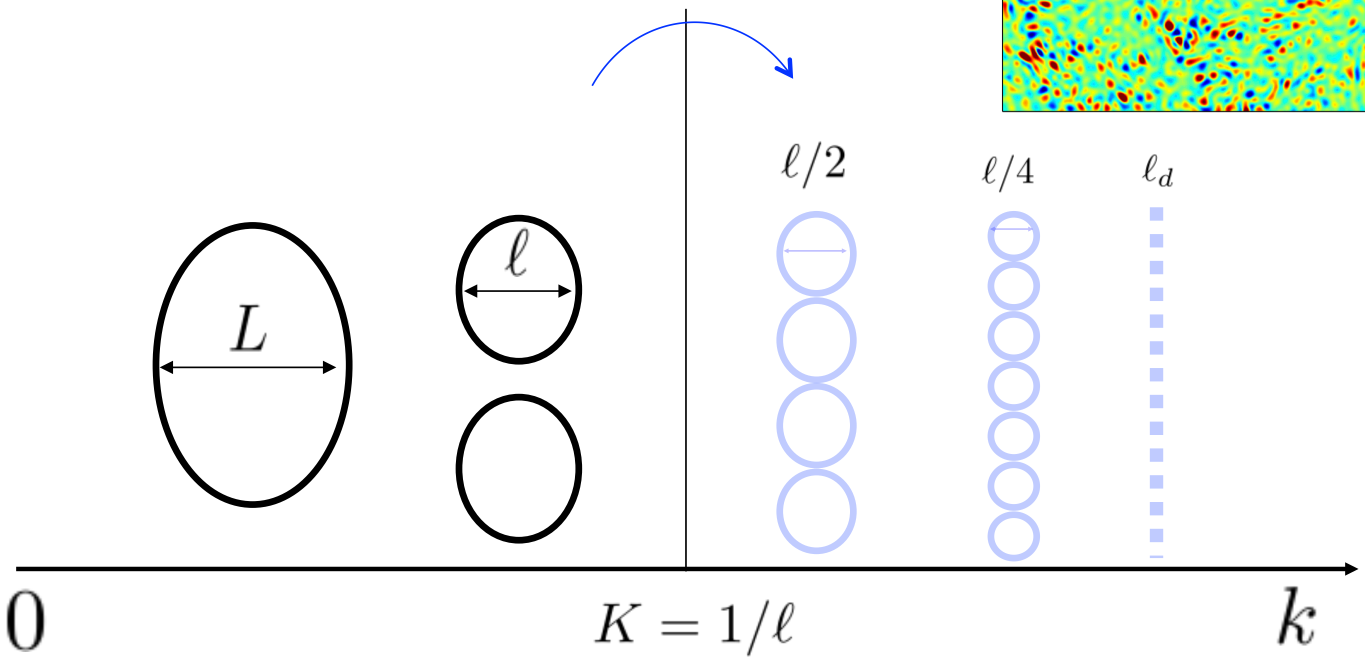
Large-scale energy budget

$$\partial_t \frac{|\overline{\mathbf{u}}|^2}{2} + \nabla \cdot [\cdots] = -\Pi_\ell^E - \nu |\nabla \overline{\mathbf{u}}|^2$$

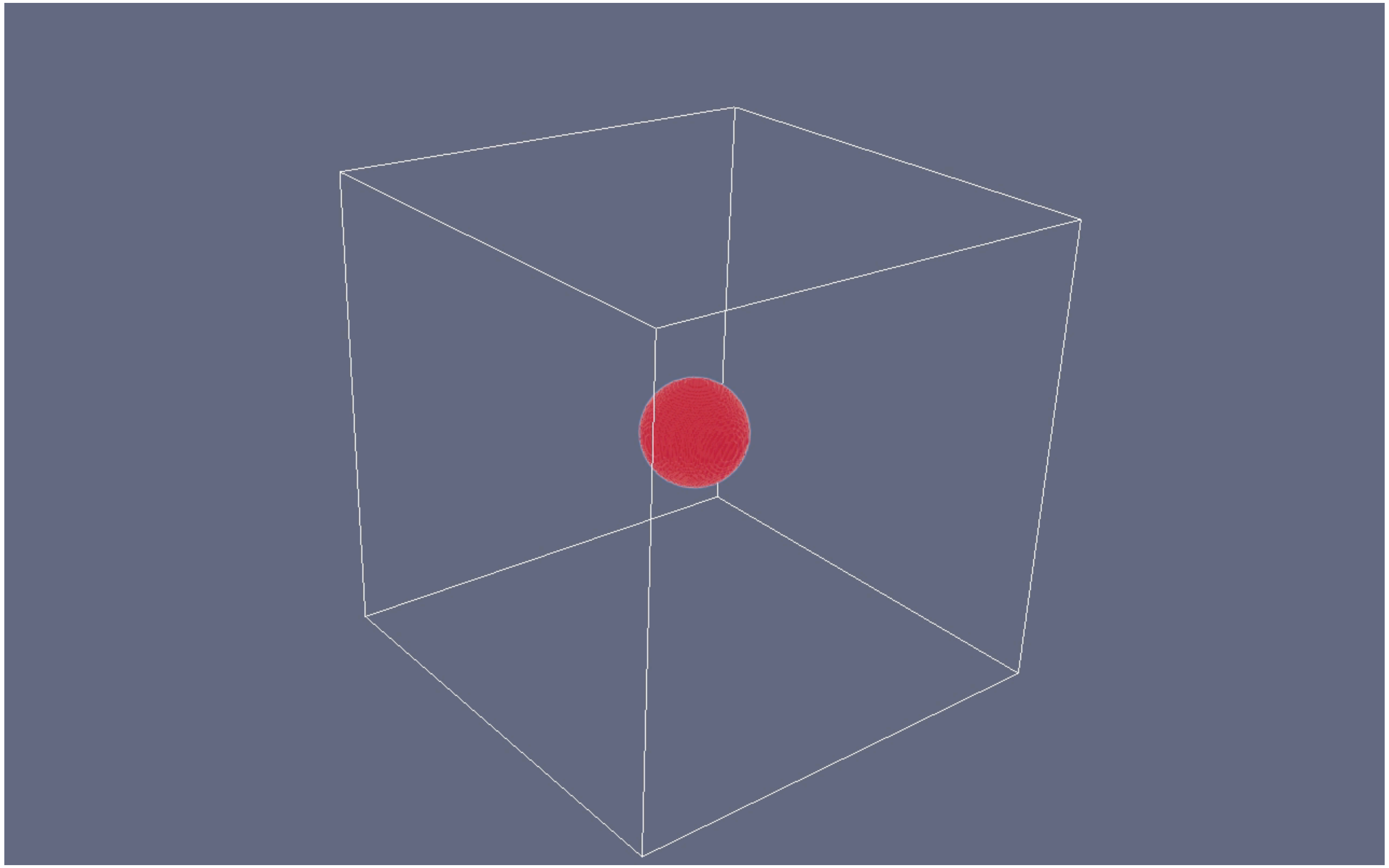
$$\boxed{\Pi_\ell^E = -\nabla \overline{\mathbf{u}}_\ell : \boldsymbol{\tau}_\ell}$$

Energy flux





Conflating cascade with spatial transport



Simulation of a passive tracer in a 3D turbulent flow
by Aarne Lees (PhD student, U. of Rochester)

Measuring Energy Transfer

Large-scale energy budget

$$\partial_t \frac{|\overline{\mathbf{u}}|^2}{2} + \boldsymbol{\nabla}\cdot[\cdots] = -\Pi_\ell^E - \nu|\boldsymbol{\nabla}\overline{\mathbf{u}}|^2$$

Subgrid scale (SGS) flux

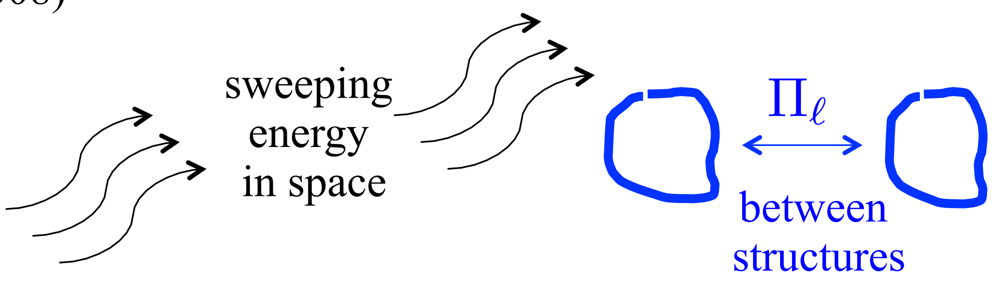
$$\Pi_\ell^E(\mathbf{x}) = -\partial_j \overline{u}_i \left[\overline{u_i u_j} - \overline{u}_i \, \overline{u}_j\right]$$

Frisch (1995), Scott & Wang (2005), Tulloch, Marshall, and Smith (2011)

$$\Pi_\ell(\mathbf{x}) = \overline{u}_i \; u_j \; \partial_j (u_i - \overline{u}_i)$$

Lindborg (2006), Mininni, et al. (2008)

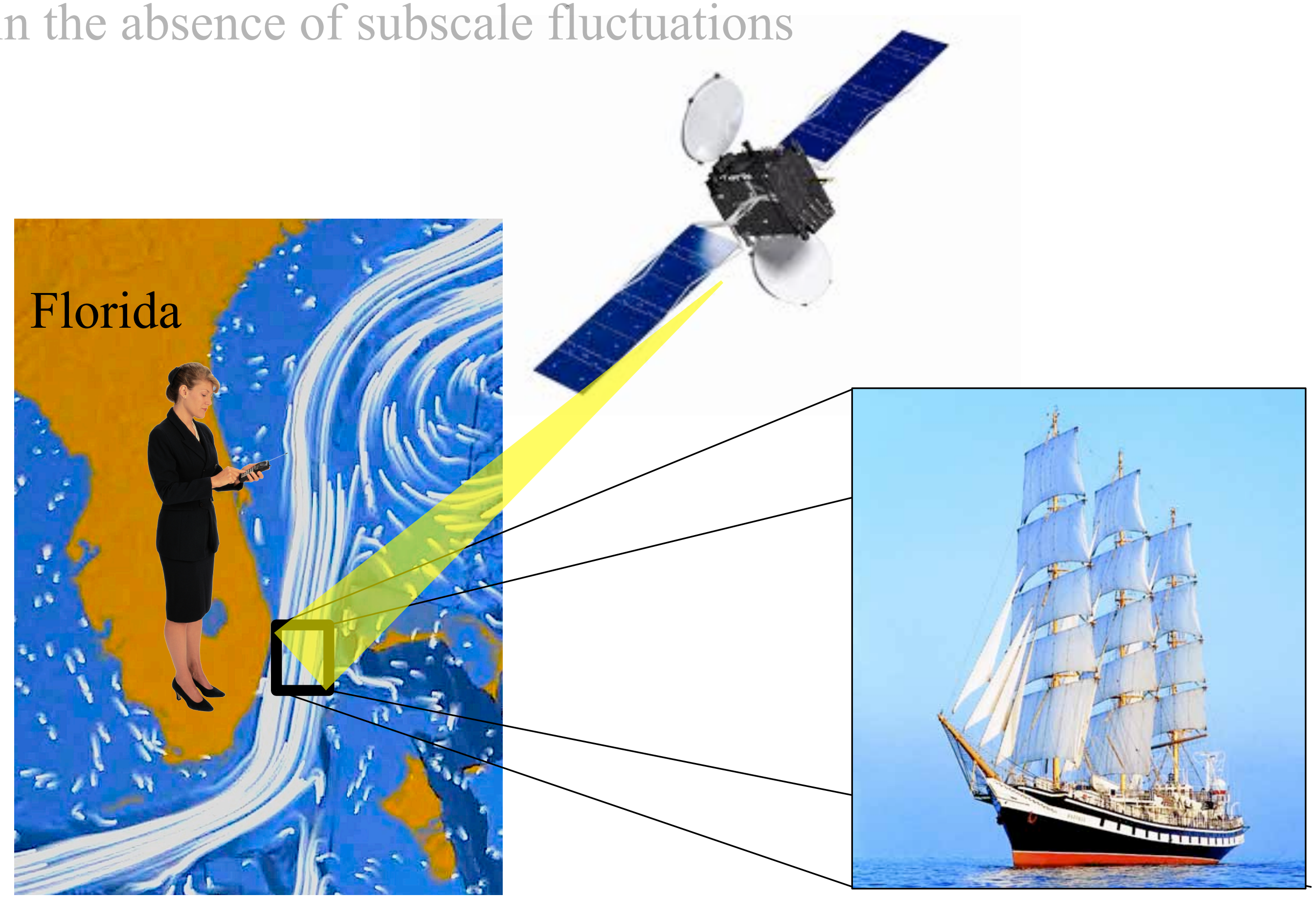
$$\Pi_\ell(\mathbf{x}) = \overline{u}_i \; \partial_j (\overline{u_j u_i})$$



Measuring Energy Transfer

Any measure of the energy exchange must satisfy:

1. Galilean Invariance [Speziale 85; Germano 92; Eyink 94]
2. Vanish in the absence of subscale fluctuations



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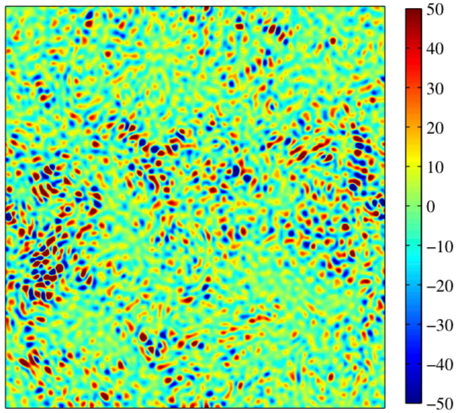
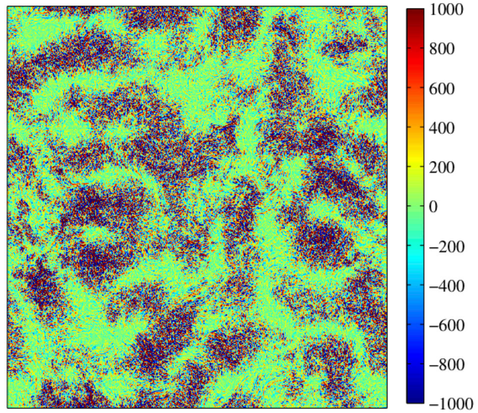
[Aluie & Kurien 11]

SGS definition





Frisch (1995) definition

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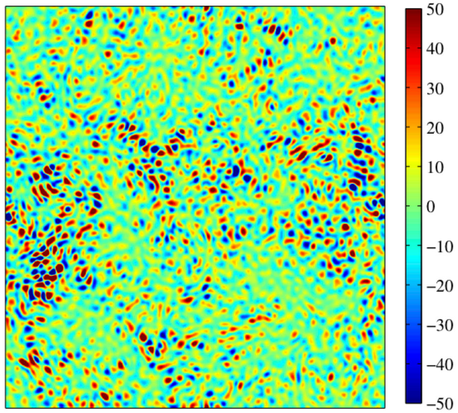
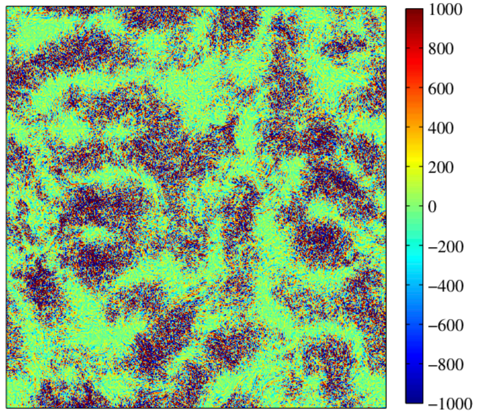
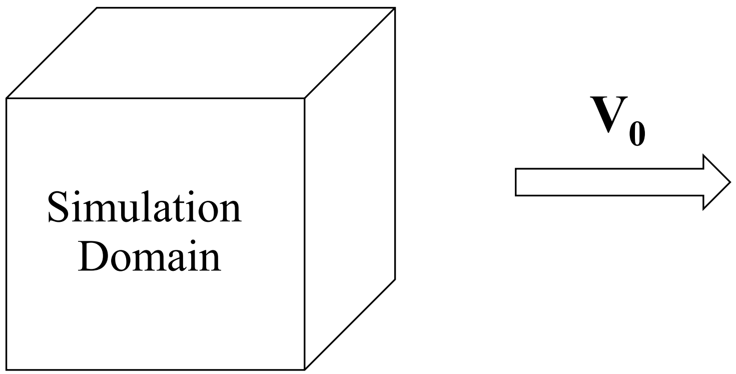




Frisch (1995) definition



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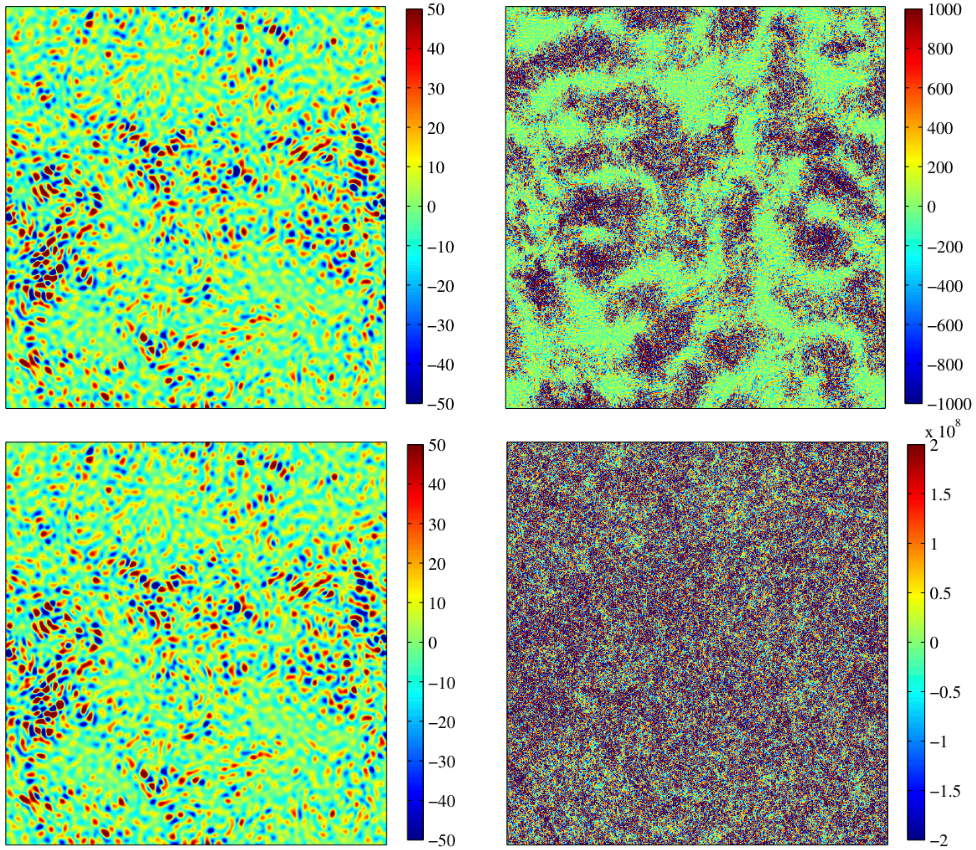
SGS definition

Frisch (1995) definition



After boosting

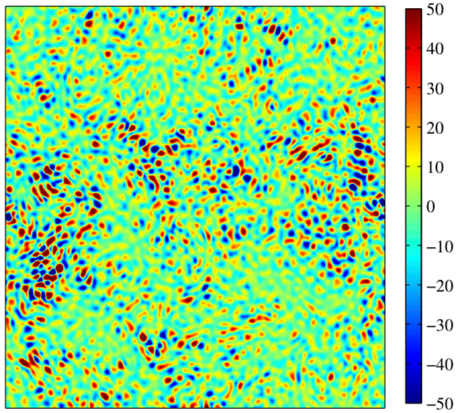
After boosting

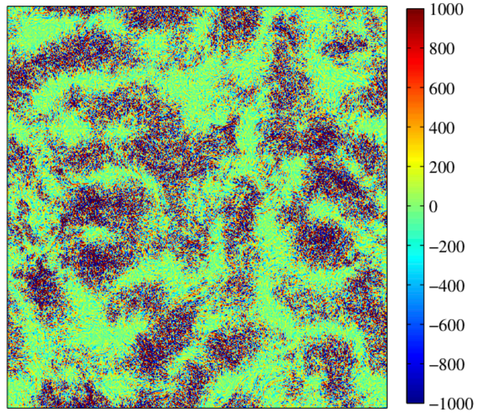
Measuring Energy Transfer

Any measure of the energy exchange must satisfy:

1. ~~Galilean Invariance~~ A more general property [Aluie & Eyink 09,10]
2. Vanish in the absence of subscale fluctuations

SGS definition





Frisch (1995) definition

Contains all information on the multi-scale *structures involved.*

Measuring Energy Transfer

Any measure of the energy exchange must satisfy:

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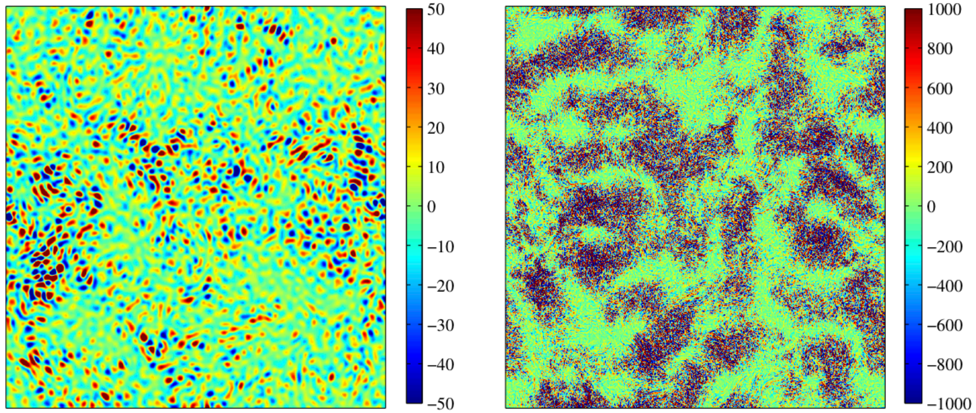
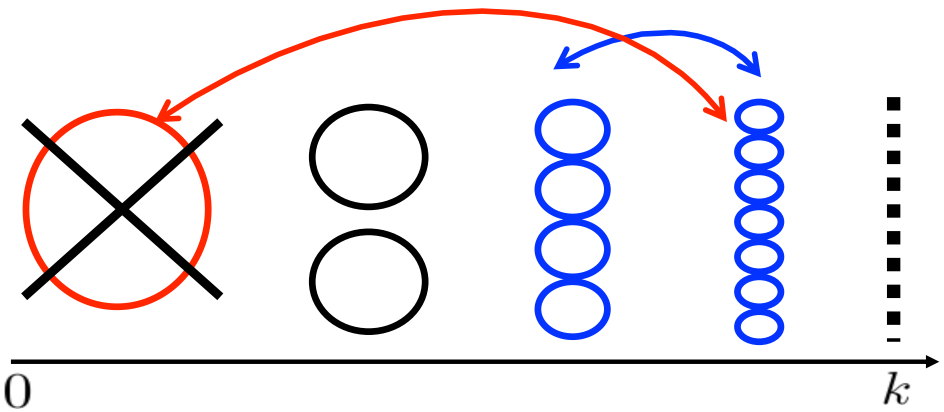
SGS definition



Frisch (1995) definition

Negligible Role



Measuring Energy Transfer

Any measure of the energy exchange must satisfy:

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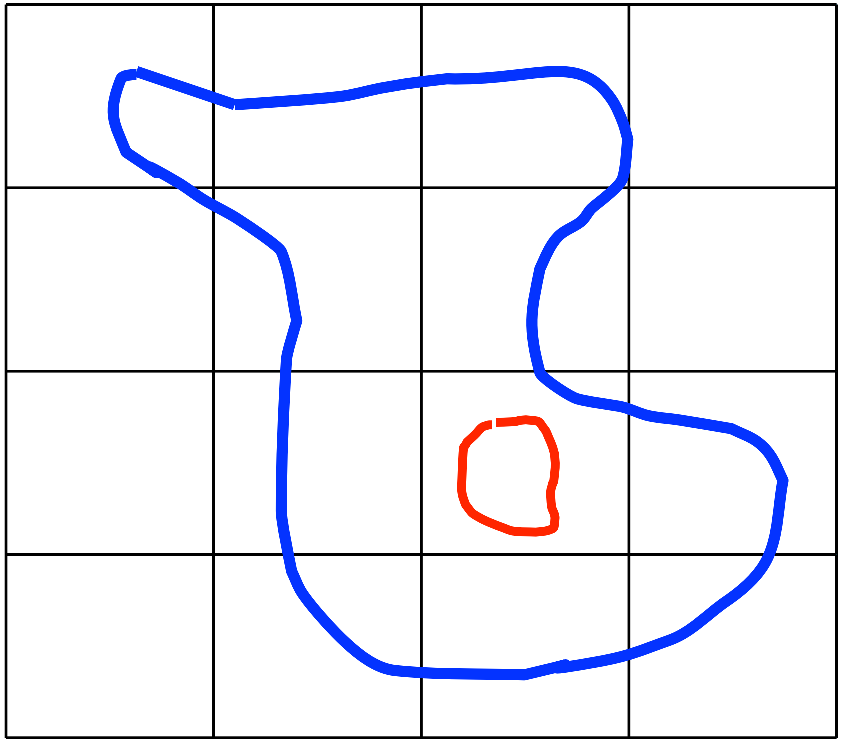

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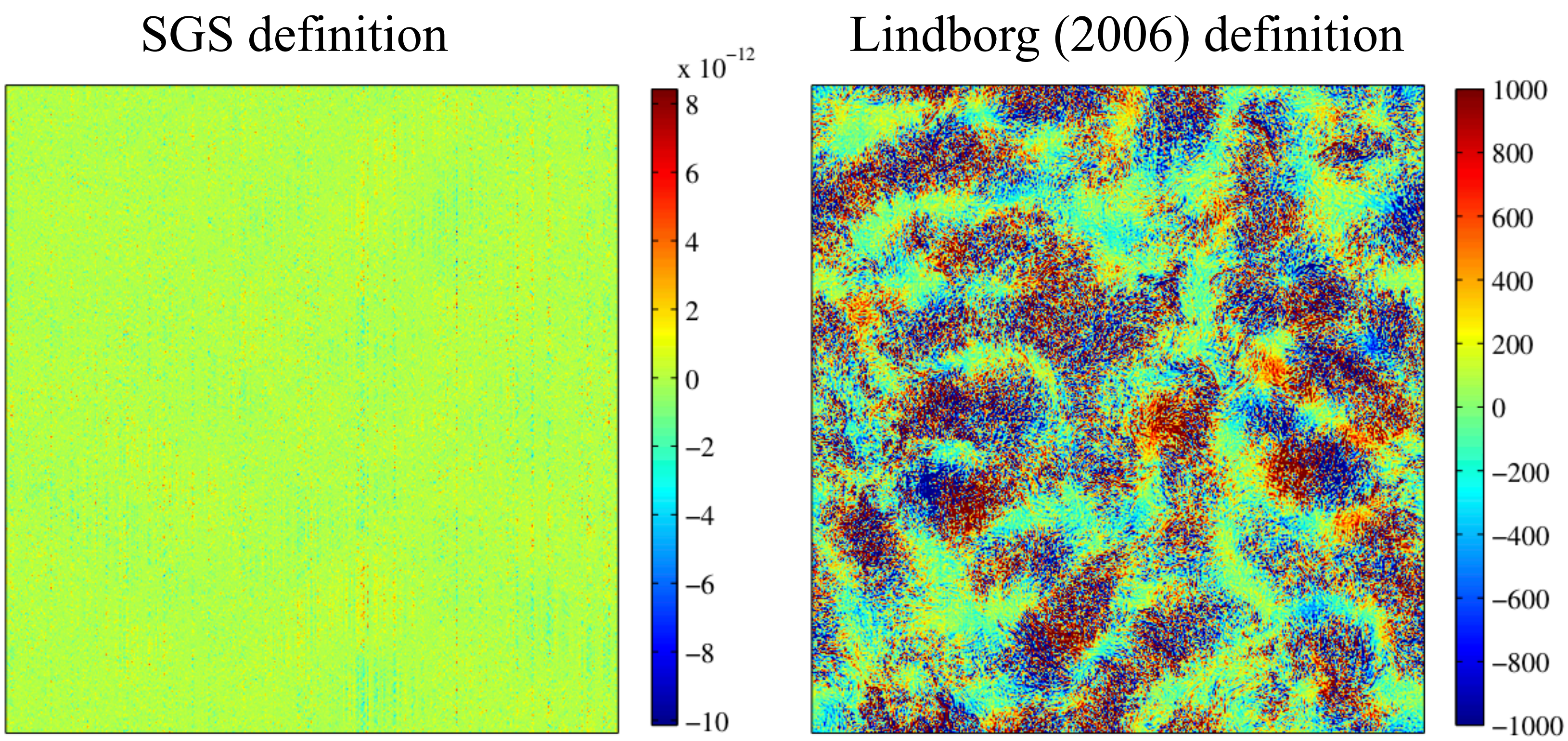

Cascade of Energy

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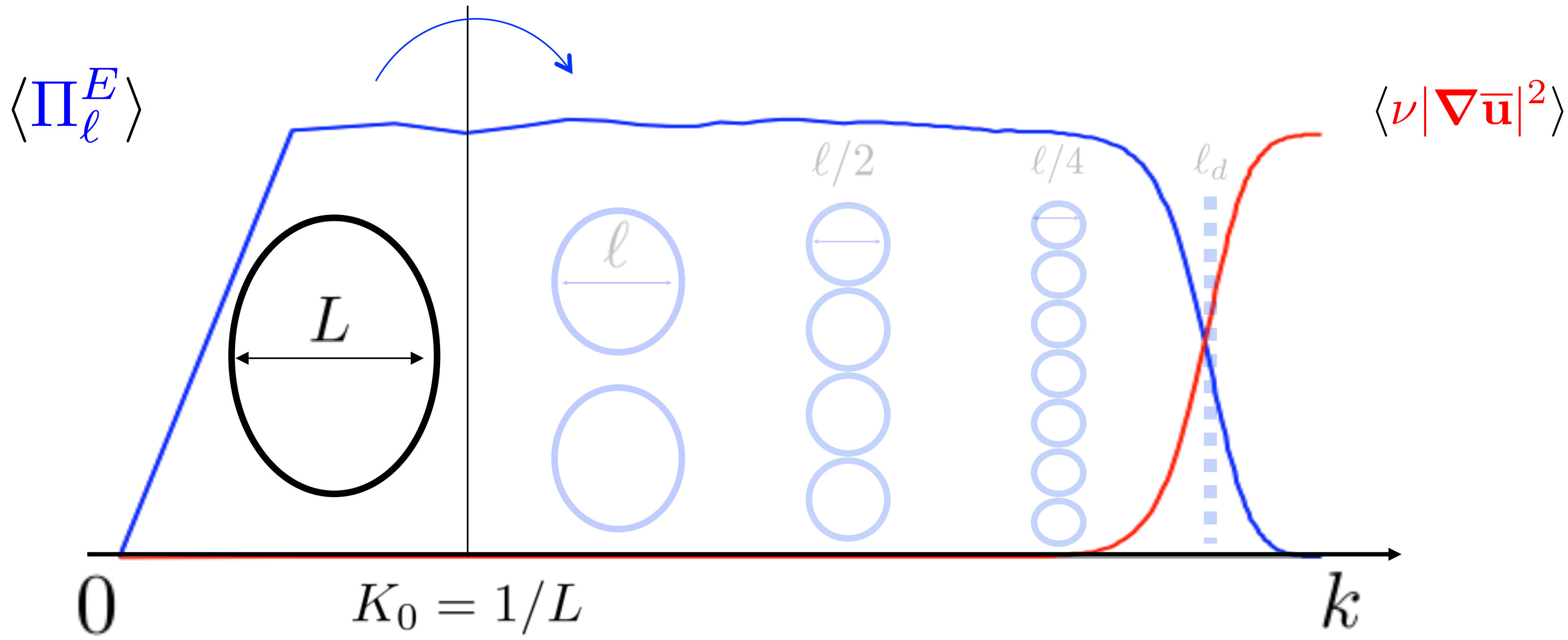
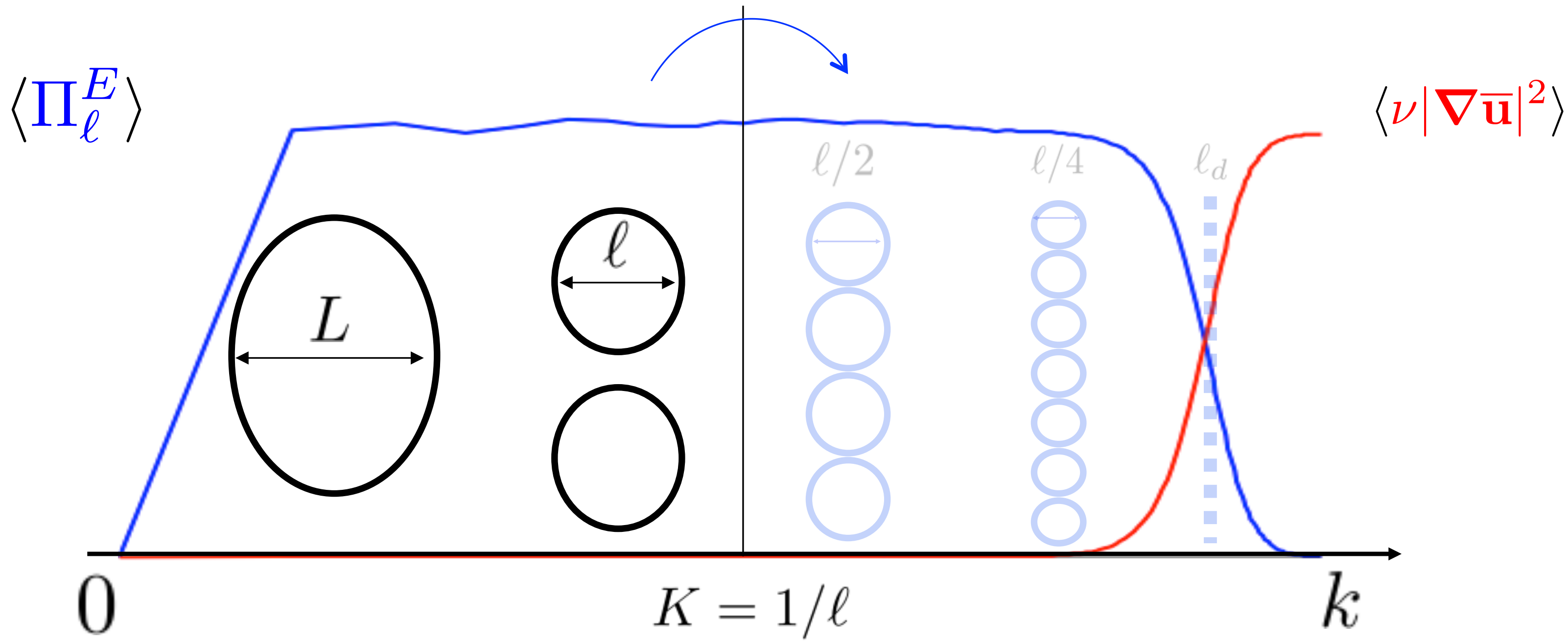
$$\partial_t \frac{|\overline{\mathbf{u}}|^2}{2} + \boldsymbol{\nabla} \cdot [\cdots] = -\Pi_\ell^E \; -\nu|\boldsymbol{\nabla}\overline{\mathbf{u}}|^2$$



Cascade of Energy

Large-scale energy budget

$$\partial_t \frac{|\overline{\mathbf{u}}|^2}{2} + \boldsymbol{\nabla} \cdot [\cdots] = -\Pi_\ell^E \quad -\nu|\boldsymbol{\nabla}\overline{\mathbf{u}}|^2$$



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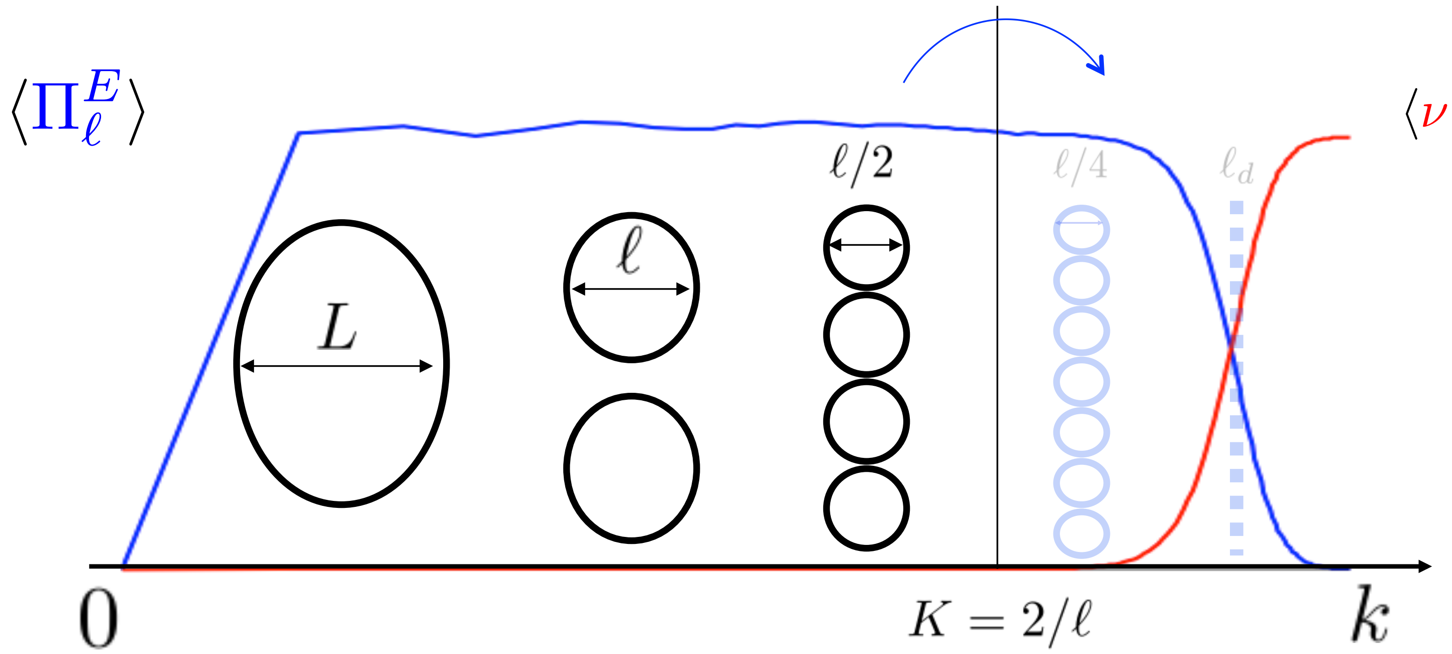

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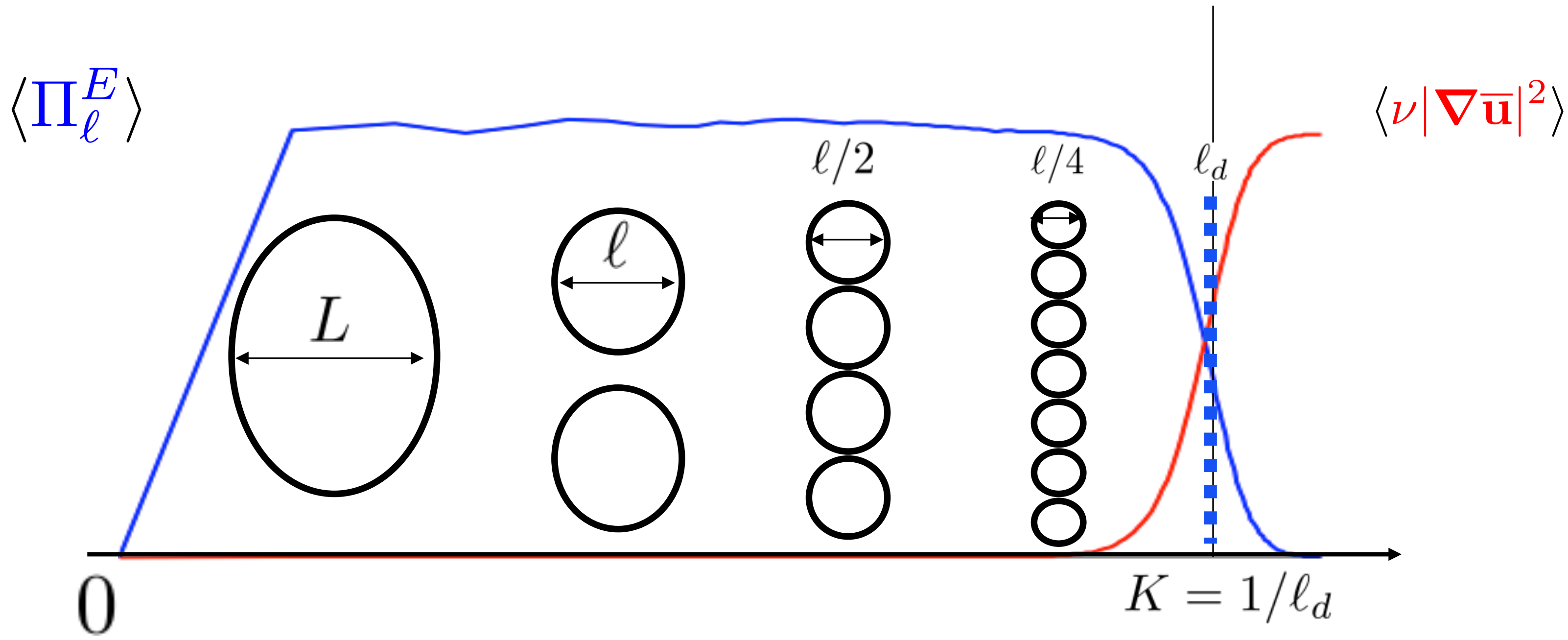
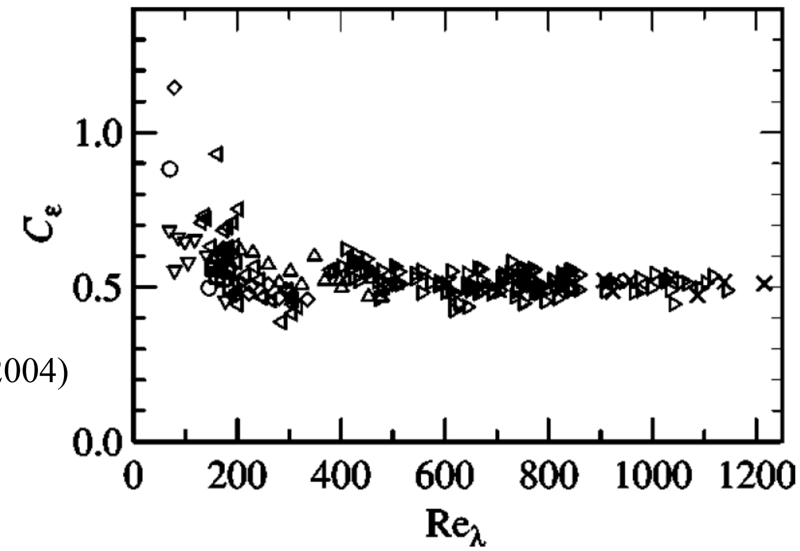


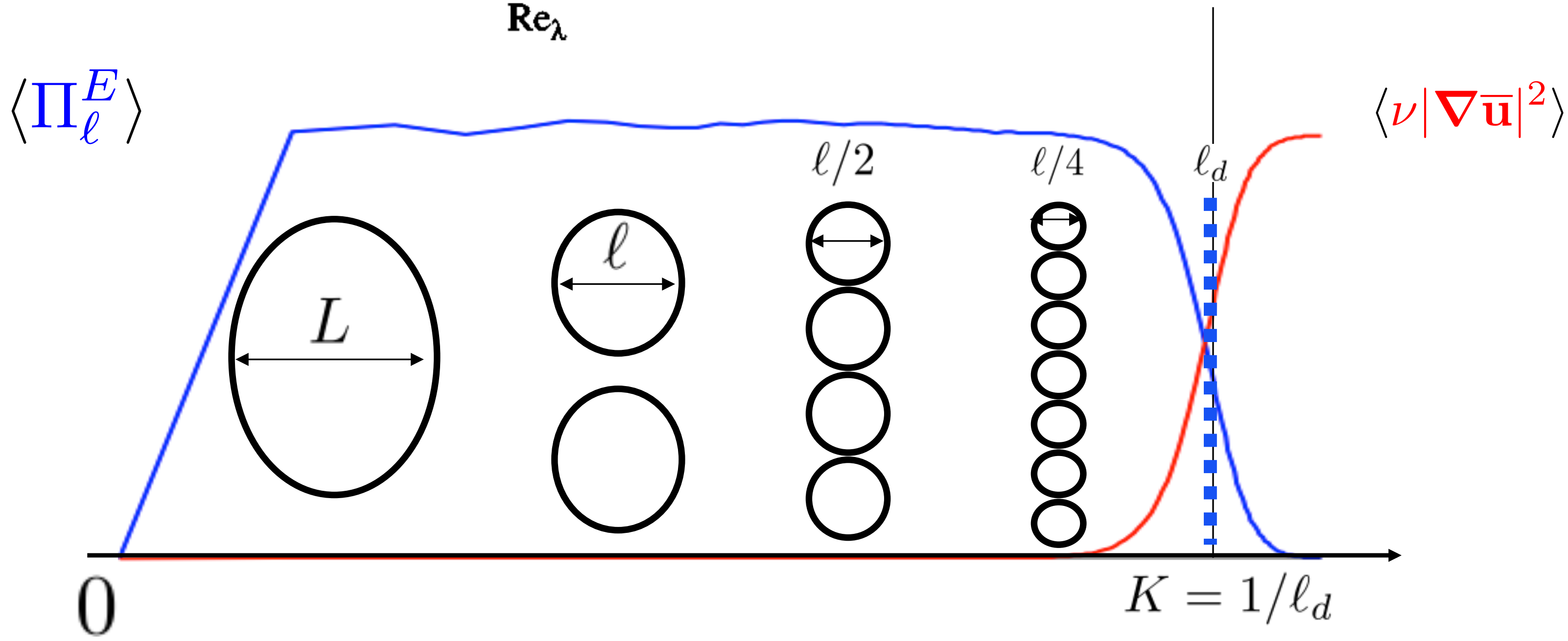
Zeroth Law of Hydrodynamic Turbulence



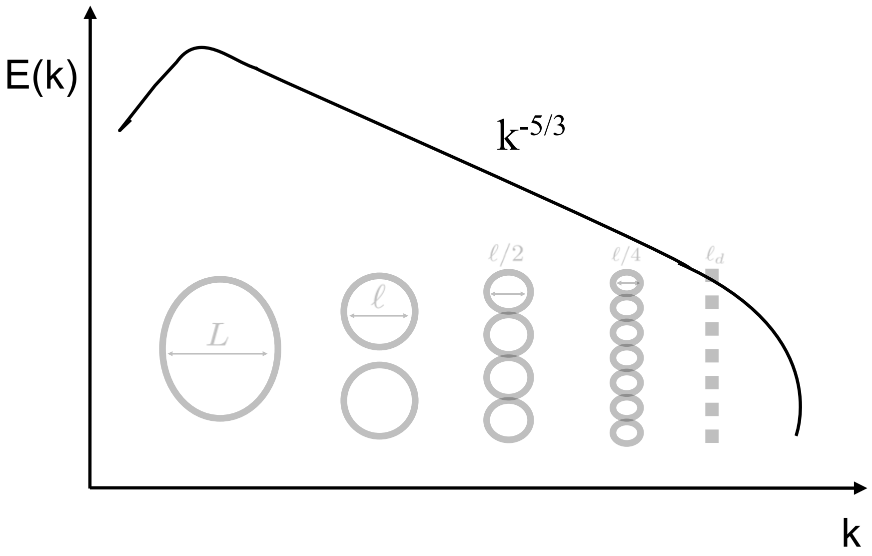
Pearson et al.,
Phys. Rev. E (2004)

$$C_\varepsilon = \frac{\varepsilon_{dissip}}{U_{rms}^3/L}$$

G. I. Taylor,
Proc. Roy. Soc. Lond., (1935)

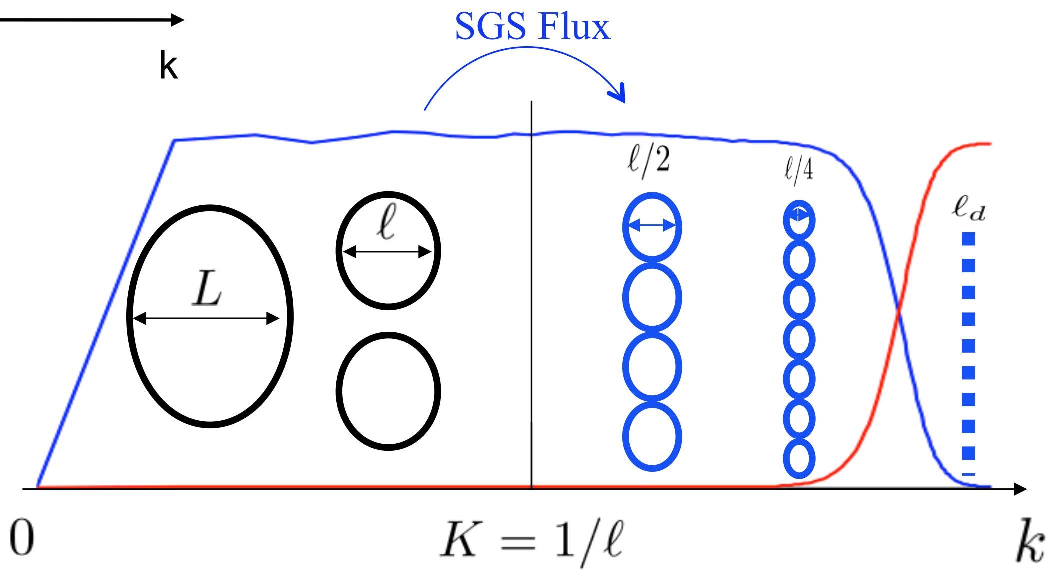


Kolmogorov's Theory



Critical Features

1. Viscosity localized to smallest scales
2. Scale-independent energy flux (4/5th law)
3. Scale locality

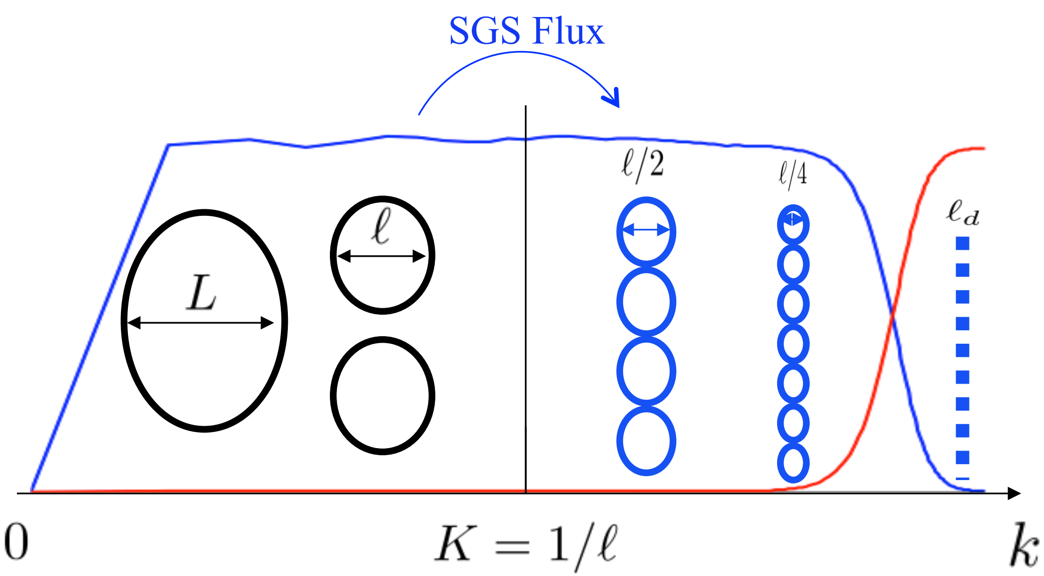


Incompressible Turbulence

$$\partial_t \frac{|\overline{\mathbf{u}}|^2}{2} + \boldsymbol{\nabla}\cdot[\cdots] = -\Pi_\ell^E \; -\nu|\boldsymbol{\nabla}\overline{\mathbf{u}}|^2$$

<u>Goal I</u>

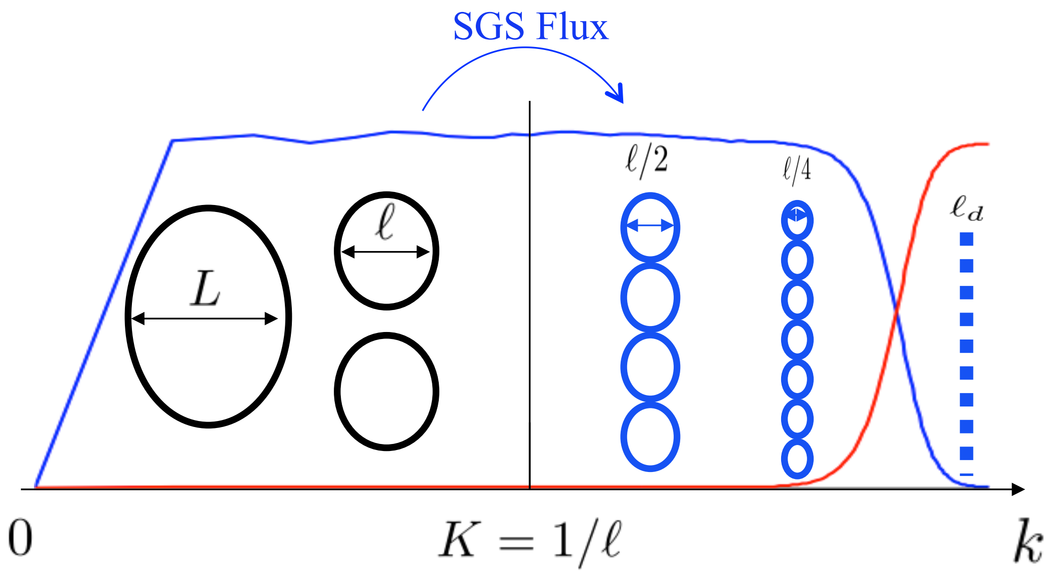
Viscosity localized
to smallest scales



Incompressible Turbulence

$$\partial_t \frac{|\overline{\mathbf{u}}|^2}{2} + \boldsymbol{\nabla}\cdot[\cdots] = -\Pi_\ell^E \; -\nu|\boldsymbol{\nabla}\overline{\mathbf{u}}|^2$$

$$\sum_{|\mathbf{k}|<K} \frac{d}{dt}\frac{|\widehat{\mathbf{u}}(\mathbf{k})|^2}{2} = -\Pi(K) \; - \sum_{|\mathbf{k}|<K} \nu|\widehat{\boldsymbol{\nabla}\mathbf{u}}(\mathbf{k})|^2$$



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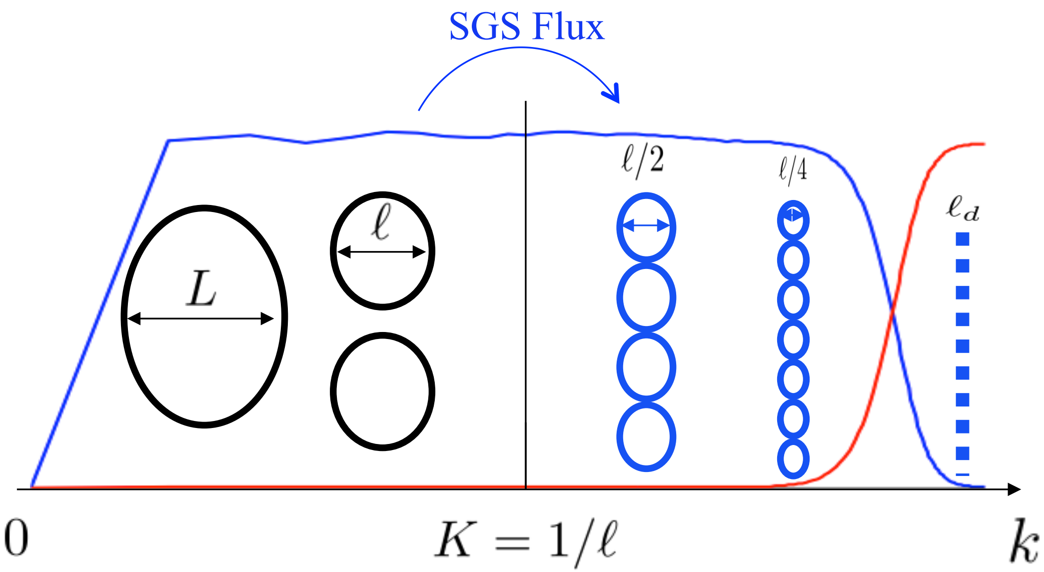

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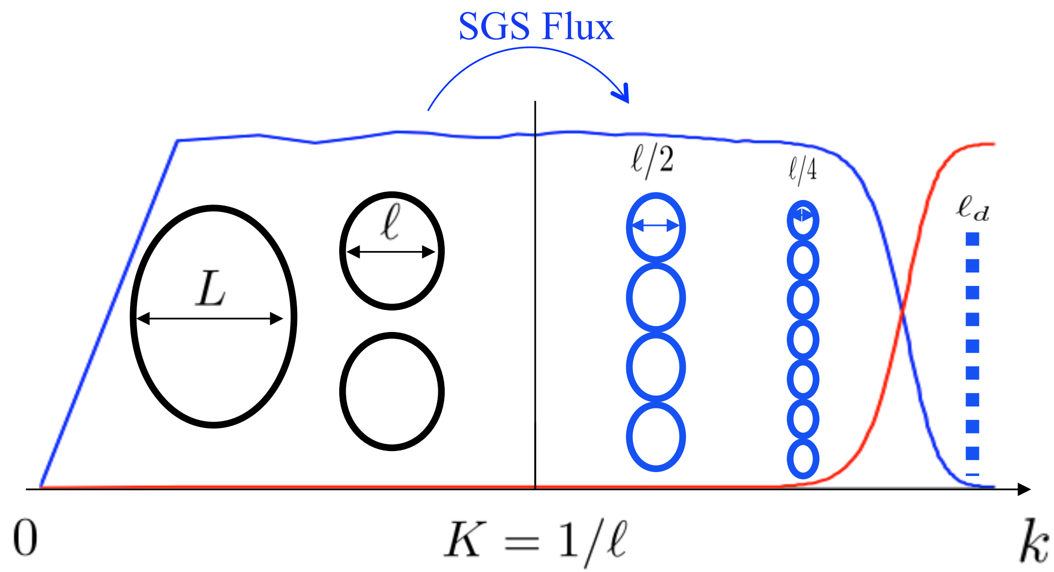
$$\sim \nu\,|\mathbf{k}|^2|\widehat{\mathbf{u}}(\mathbf{k})|^2$$



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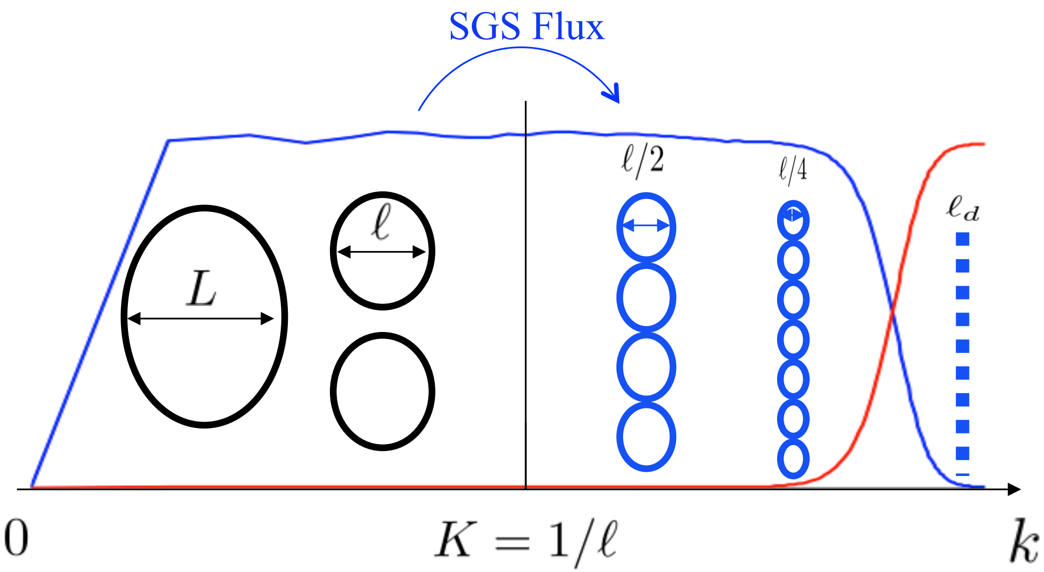

Incompressible Turbulence

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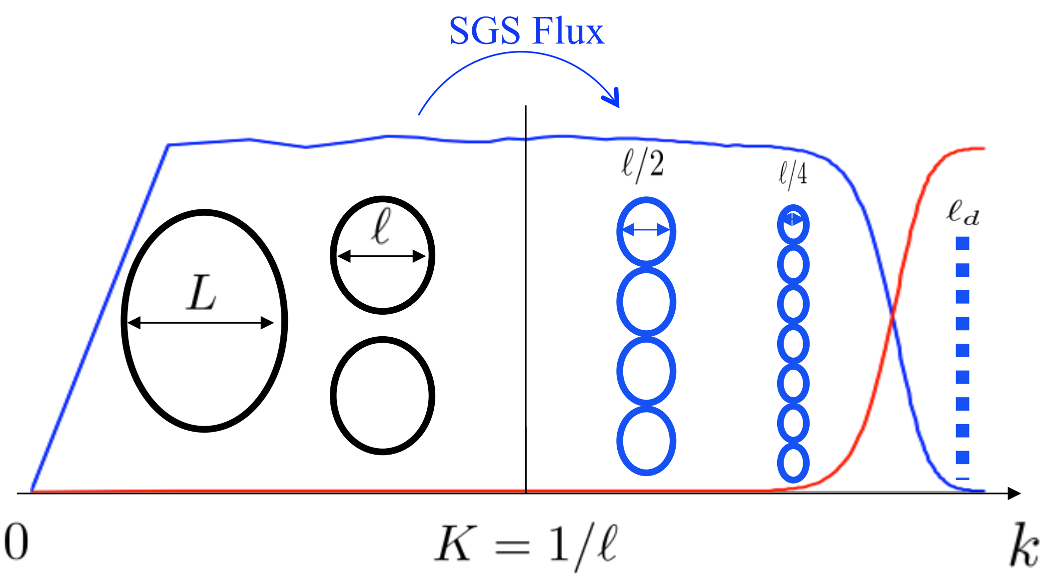
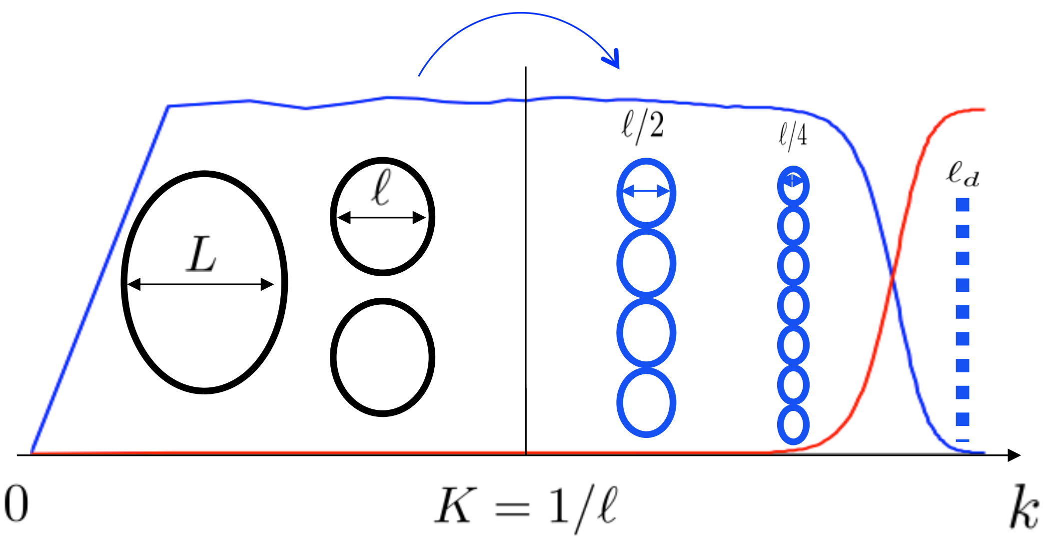
$$\sim \nu\,|\mathbf{k}|^2|\widehat{\mathbf{u}}(\mathbf{k})|^2$$

purely kinematic



Compressible Turbulence

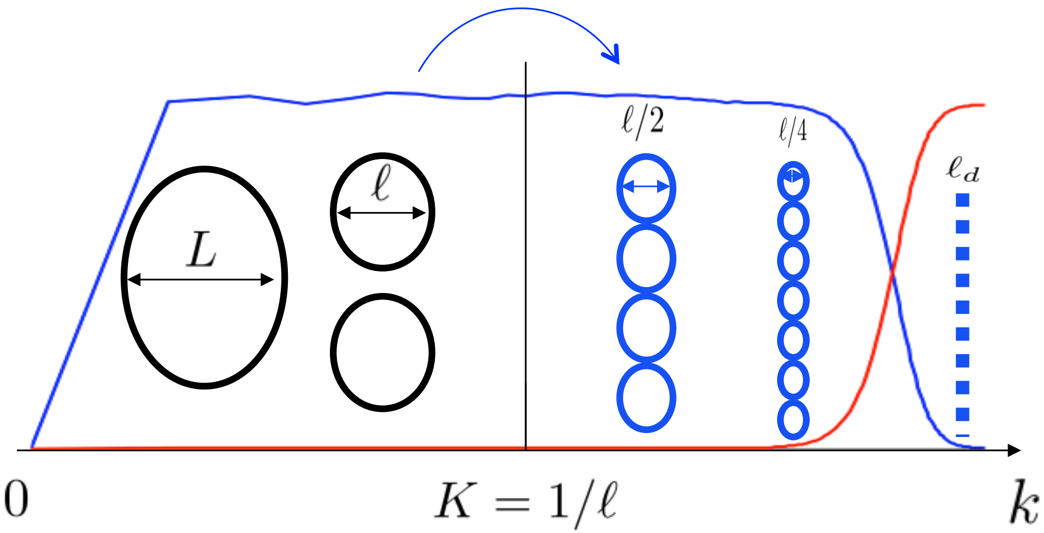
$$\partial_t \frac{\rho|\mathbf{u}|^2}{2}$$



Compressible Turbulence

$$\sum_{|\mathbf{k}|<K} \frac{1}{2}\frac{d}{dt}|\widehat{\sqrt{\rho}\mathbf{u}}(\mathbf{k})|^2 \longleftrightarrow \frac{1}{2}\left|\overline{\sqrt{\rho}\mathbf{u}}\right|^2$$

[Kida & Orszag (1990)]



Compressible Turbulence

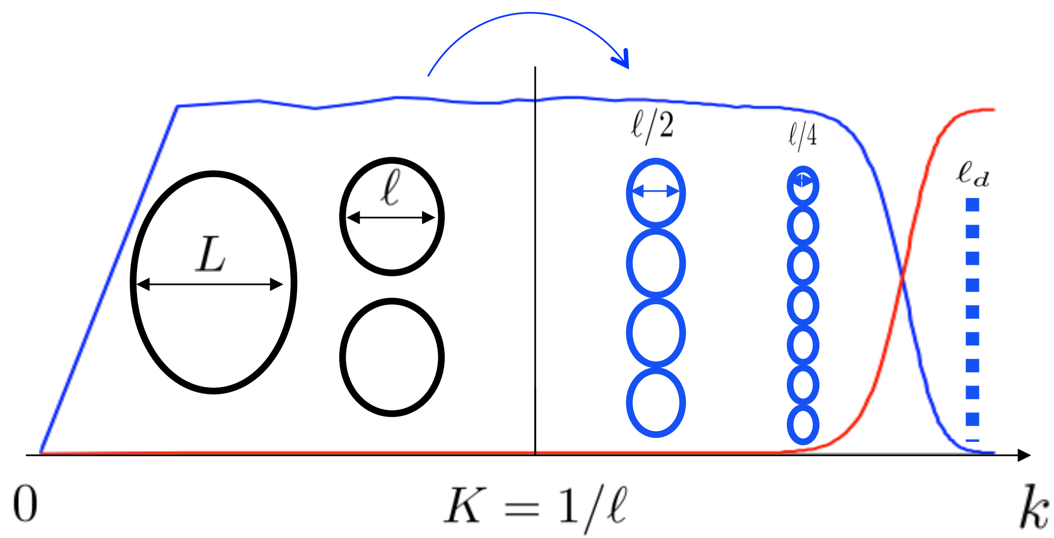
$$\sum_{|\mathbf{k}|<K} \frac{1}{2}\frac{d}{dt}|\widehat{\sqrt{\rho}\mathbf{u}}(\mathbf{k})|^2 \longleftrightarrow \frac{1}{2}\left|\overline{\sqrt{\rho}\mathbf{u}}\right|^2$$

[Kida & Orszag (1990)]

OR

$$\sum_{|\mathbf{k}|,|\mathbf{q}|<K} \frac{1}{2}\frac{d}{dt}\,\widehat{\rho}(-\mathbf{k}-\mathbf{q})\;\widehat{\mathbf{u}}(\mathbf{k})\cdot\widehat{\mathbf{u}}(\mathbf{q}) \longleftrightarrow \frac{1}{2}\overline{\rho}\left|\overline{\mathbf{u}}\right|^2$$

[Chassaing (1985)]

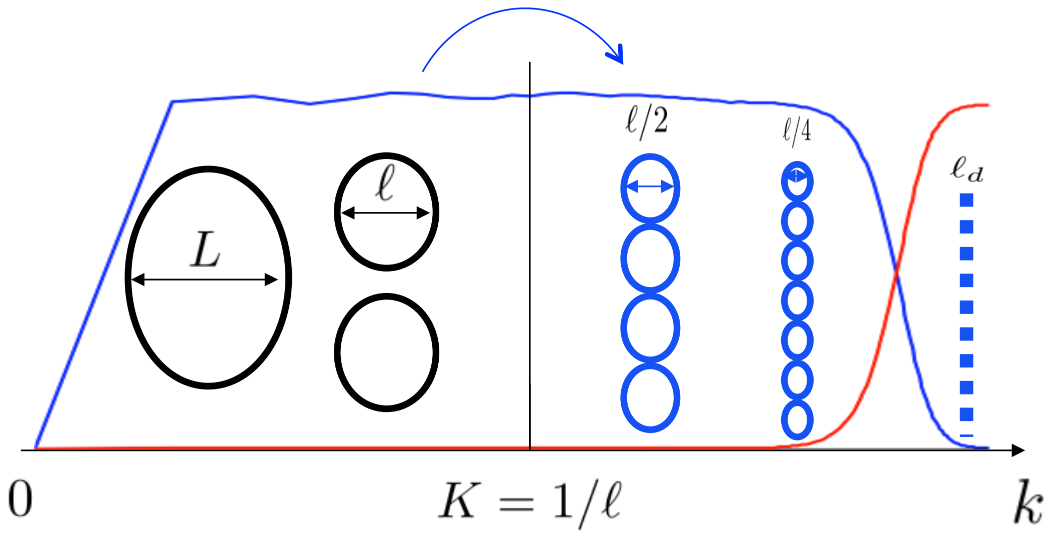


Compressible Turbulence

$$\sum_{|\mathbf{k}|<K} \frac{1}{2}\frac{d}{dt}|\widehat{\sqrt{\rho}\mathbf{u}}(\mathbf{k})|^2 = [\ldots] \quad + \text{ viscous dissipation}$$

OR

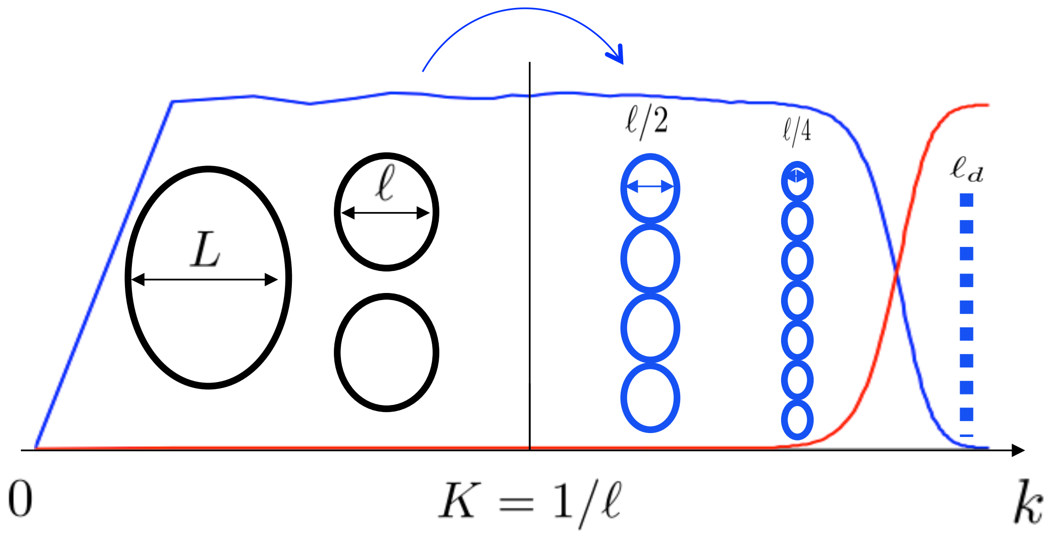
$$\sum_{|\mathbf{k}|,|\mathbf{q}|<K} \frac{1}{2}\frac{d}{dt}\,\widehat{\rho}(-\mathbf{k}-\mathbf{q})\;\widehat{\mathbf{u}}(\mathbf{k})\cdot\widehat{\mathbf{u}}(\mathbf{q}) = [\ldots] \quad + \text{ viscous dissipation}$$



Compressible Turbulence

$$\sum_{|\mathbf{k}|<K} \frac{1}{2}\frac{d}{dt}|\widehat{\sqrt{\rho}\mathbf{u}}(\mathbf{k})|^2 \qquad \text{OR} \qquad \sum_{|\mathbf{k}|,|\mathbf{q}|<K} \frac{1}{2}\frac{d}{dt}\ \widehat{\rho}(-\mathbf{k}-\mathbf{q})\ \widehat{\mathbf{u}}(\mathbf{k})\cdot\widehat{\mathbf{u}}(\mathbf{q})$$

viscous dissipation ~ $\sum_{|\mathbf{k}|<K} \mu\ \widehat{\rho^{-1}\nabla^2\mathbf{u}}(\mathbf{k})\ \widehat{\rho\mathbf{u}}(-\mathbf{k})$

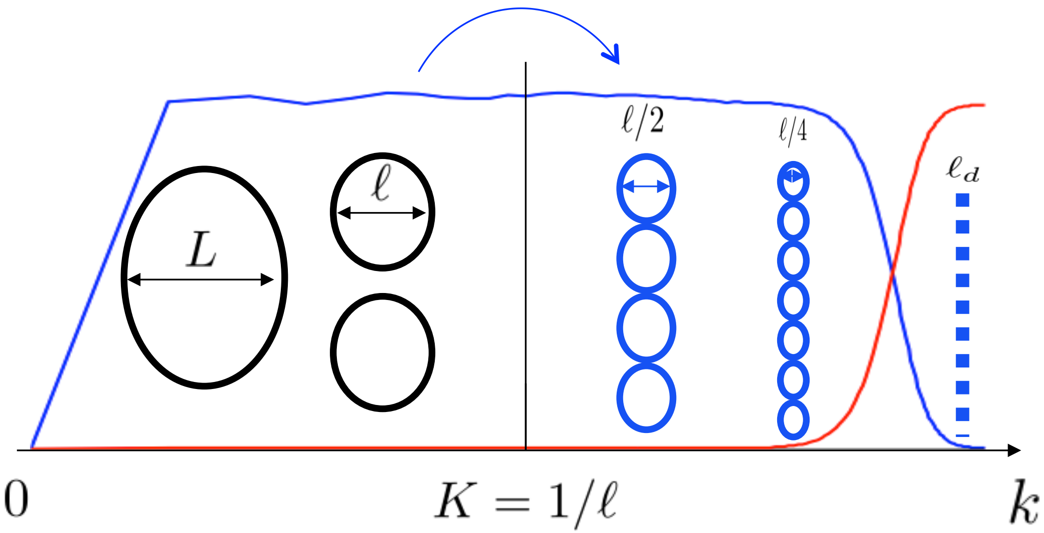


Compressible Turbulence

$$\sum_{|\mathbf{k}|<K} \frac{1}{2}\frac{d}{dt}|\widehat{\sqrt{\rho}\mathbf{u}}(\mathbf{k})|^2 \qquad \text{OR} \qquad \sum_{|\mathbf{k}|,|\mathbf{q}|<K} \frac{1}{2}\frac{d}{dt}\,\widehat{\rho}(-\mathbf{k}-\mathbf{q})\ \widehat{\mathbf{u}}(\mathbf{k})\cdot\widehat{\mathbf{u}}(\mathbf{q})$$

viscous dissipation ~ $\sum_{|\mathbf{k}|<K} \mu\,\widehat{\rho^{-1}\nabla^2\mathbf{u}}(\mathbf{k})\ \widehat{\rho\mathbf{u}}(-\mathbf{k})$

$\sim \mu|\mathbf{k}|^2$



Compressible Turbulence

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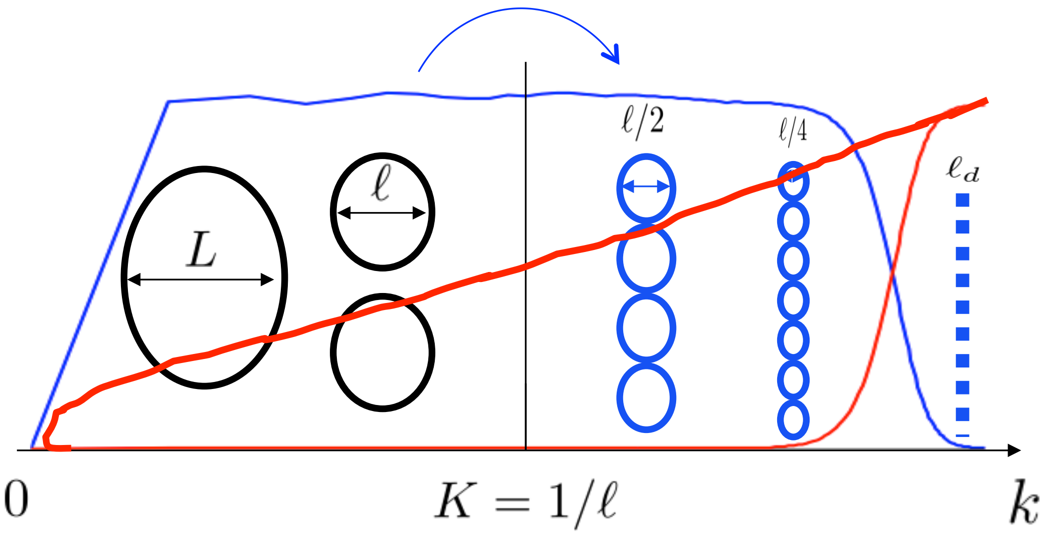


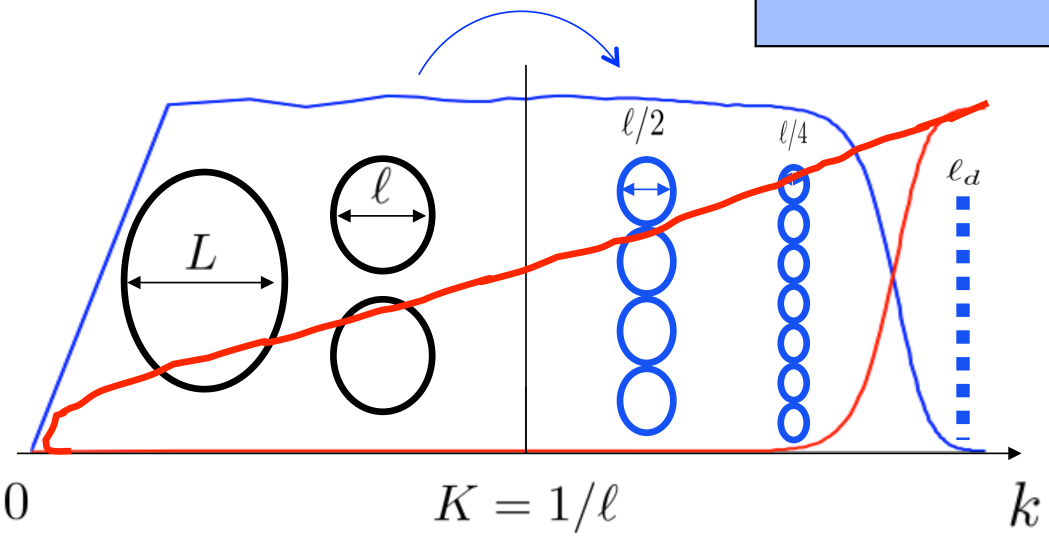
Compressible Turbulence

$$\sum_{|\mathbf{k}|<K} \frac{1}{2}\frac{d}{dt}|\widehat{\sqrt{\rho}\mathbf{u}}(\mathbf{k})|^2 \qquad \text{OR} \qquad \sum_{|\mathbf{k}|,|\mathbf{q}|<K} \frac{1}{2}\frac{d}{dt}\,\widehat{\rho}(-\mathbf{k}-\mathbf{q})\;\widehat{\mathbf{u}}(\mathbf{k})\cdot\widehat{\mathbf{u}}(\mathbf{q})$$

viscous dissipation ~ $\sum_{|\mathbf{k}|<K} \mu\,\widehat{\rho^{-1}\nabla^2\mathbf{u}}(\mathbf{k})\;\widehat{\rho\mathbf{u}}(-\mathbf{k})$

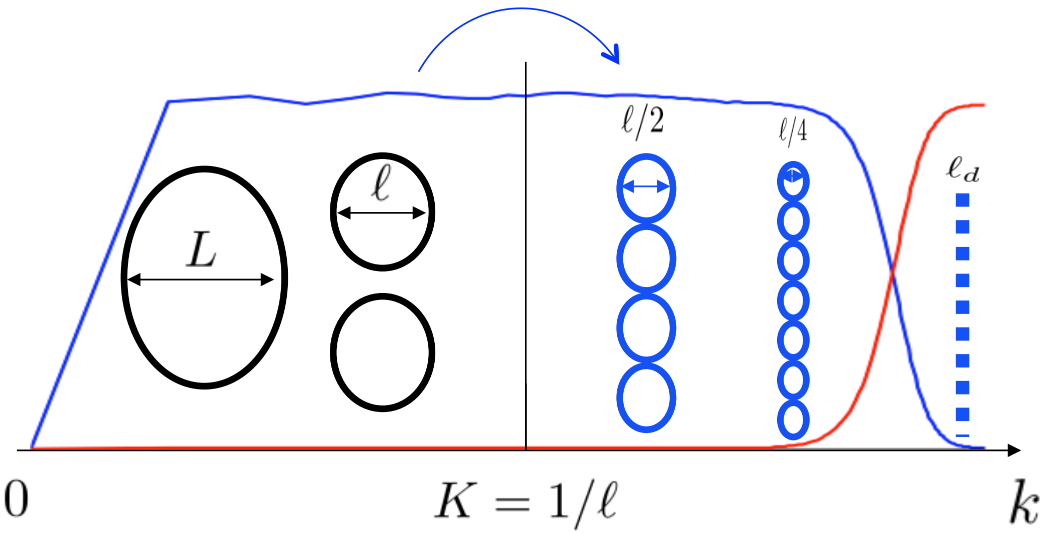
~~$\sim \mu|\mathbf{k}|^2$~~

Inviscid criterion



Compressible Turbulence

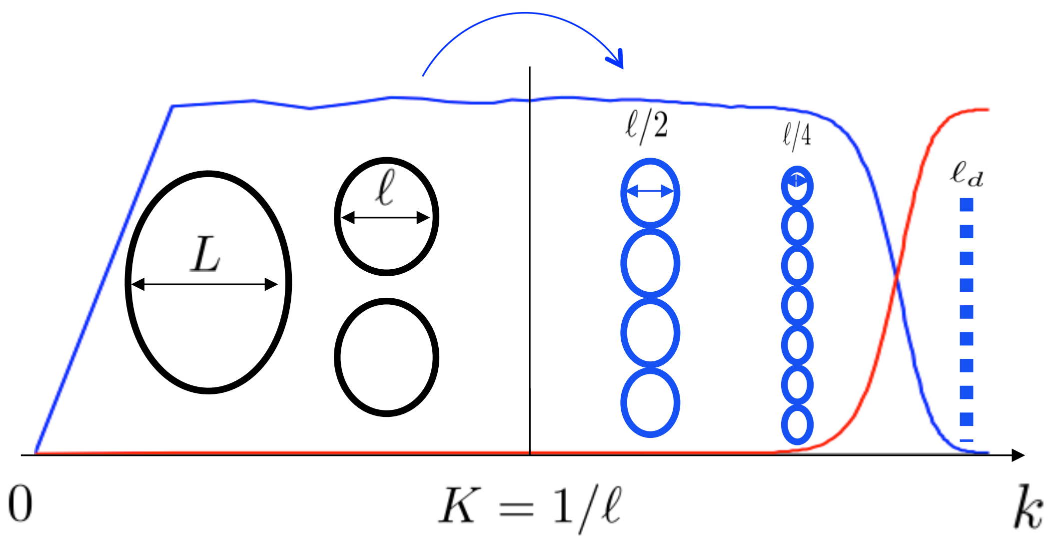
$$\frac{1}{2}\frac{d}{dt}\left\langle \frac{\sum_{|\mathbf{k}|,|\mathbf{q}|<K} \widehat{\rho\mathbf{u}}(\mathbf{k})\cdot\widehat{\rho\mathbf{u}}(\mathbf{q})\ e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}}}{\sum_{|\mathbf{p}|<K} \widehat{\rho}(\mathbf{p})\ e^{i\mathbf{p}\cdot\mathbf{x}}} \right\rangle = [\dots] \qquad + \text{ viscous dissipation}$$



Compressible Turbulence

$$\frac{1}{2}\frac{d}{dt}\left\langle \frac{\sum_{|\mathbf{k}|,|\mathbf{q}|<K} \widehat{\rho\mathbf{u}}(\mathbf{k})\cdot\widehat{\rho\mathbf{u}}(\mathbf{q})\ e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}}}{\sum_{|\mathbf{p}|<K} \widehat{\rho}(\mathbf{p})\ e^{i\mathbf{p}\cdot\mathbf{x}}} \right\rangle = [\ldots]$$

+ viscous dissipation



Compressible Turbulence

$$\frac{1}{2}\frac{d}{dt}\left\langle \frac{\sum_{|\mathbf{k}|,|\mathbf{q}|<K} \widehat{\rho\mathbf{u}}(\mathbf{k})\cdot\widehat{\rho\mathbf{u}}(\mathbf{q})\ e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}}}{\sum_{|\mathbf{p}|<K} \widehat{\rho}(\mathbf{p})\ e^{i\mathbf{p}\cdot\mathbf{x}}} \right\rangle$$

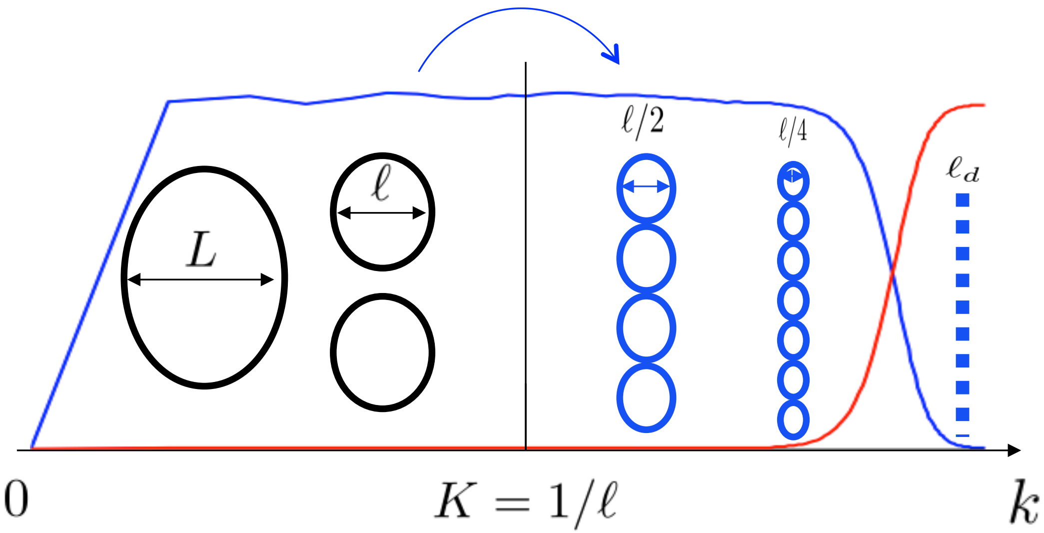
$$\frac{1}{2}\frac{d}{dt}\langle \overline{\rho}_\ell |\widetilde{\mathbf{u}}_\ell|^2 \rangle$$

Inviscid criterion

Favre Filtering (1965)

$$\widetilde{f}_\ell(\mathbf{x}) \equiv \frac{\overline{\rho f}_\ell}{\overline{\rho}_\ell}$$

$$\overline{g}_\ell(\mathbf{x}) = \sum_{|\mathbf{k}|<\ell^{-1}} \widehat{g}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$



Compressible Turbulence

$$\frac{1}{2}\frac{d}{dt}\langle \overline{\rho}_\ell |\widetilde{\mathbf{u}}_\ell|^2\rangle$$

Inviscid criterion

Favre Filtering (1965)

$$\widetilde{f}_\ell(\mathbf{x}) \equiv \frac{\overline{\rho f}_\ell}{\overline{\rho}_\ell}$$

Aluie H., *Physica D* (2013)

Proposition 1. *For a constant viscosity, $\mu(\mathbf{x}) = \mu$, if velocity solutions $\mathbf{u}$ of the compressible Navier-Stokes equation (1)-(3) over a domain $\mathbb{T}^d$ have finite 2nd-order moments: $\int_{\mathbb{T}^d} d\mathbf{x}|\mathbf{u}|^2 < \infty$, then viscous terms in the large-scale momentum eq. (8) vanish pointwise as $\mu \to 0$.*

$$\begin{aligned}\left|\mu\, \partial_j\partial_i \overline{\mathbf{u}}_\ell(\mathbf{x})\right| &\leq \frac{\mu}{\ell^2}\int d\mathbf{r}\left|(\partial_j\partial_i G)_\ell(\mathbf{r})\ \mathbf{u}(\mathbf{x}+\mathbf{r})\right| \\ &\leq \frac{\mu}{\ell^2} V^{\frac{1}{p}} \left\|(\partial_j\partial_i G)_\ell\right\|_p V^{\frac{1}{q}}\left\|\mathbf{u}\right\|_q = \frac{\mu}{\ell^2}\left(\frac{L_{\text{dom}}}{\ell}\right)^{d(1-\frac{1}{p})}\left\|\mathbf{u}\right\|_q\left(\int ds\left|\frac{\partial^2 G(\mathbf{s})}{\partial s_i \partial s_j}\right|^p\right)^{\frac{1}{p}}\end{aligned}$$

Compressible Turbulence

$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle$$

Inviscid criterion

Favre Filtering (1965)

$$\widetilde{f}_\ell(\mathbf{x}) \equiv \frac{\overline{\rho f}_\ell}{\overline{\rho}_\ell}$$

Aluie H., *Physica D* (2013)

Proposition 2. *For a constant $\mu(\mathbf{x}) = \mu$, if solutions $(\rho, \mathbf{u})$ of the compressible Navier-Stokes equation (1)-(3) over domain $\mathbb{T}^d$ have finite 3rd-order moments: $\int_{\mathbb{T}^d} d\mathbf{x}|\rho|^3 < \infty$ and $\int_{\mathbb{T}^d} d\mathbf{x}|\mathbf{u}|^3 < \infty$, and finite mean specific volume, $\int_{\mathbb{T}^d} d\mathbf{x}\ \rho^{-1} < \infty$, then for positive kernels $G(\mathbf{r}) \geq 0$, viscous terms in the large-scale kinetic energy budget (13) vanish pointwise as $\mu \to 0$.*

$$\mu|\nabla\tilde{\mathbf{u}}\nabla\overline{\mathbf{u}}| \leq \frac{\mu}{\ell^2}\|\mathbf{u}\|_3^2\left[A(L_{\rm dom}/\ell)+B(L_{\rm dom}/\ell)\frac{\|\rho\|_3}{\overline{\rho}}+C(L_{\rm dom}/\ell)\frac{\|\rho\|_3^2}{\overline{\rho}^2}\right]A(L_{\rm dom}/\ell).$$

Factors $1/\overline{\rho}_\ell(\mathbf{x})$ in the above expression are finite because $1/\rho$ is a convex function of density over $\rho \in [0,\infty)$. When $G(\mathbf{r}) \geq 0$, coarse-graining is an averaging operation and we can use Jensen's inequality to obtain

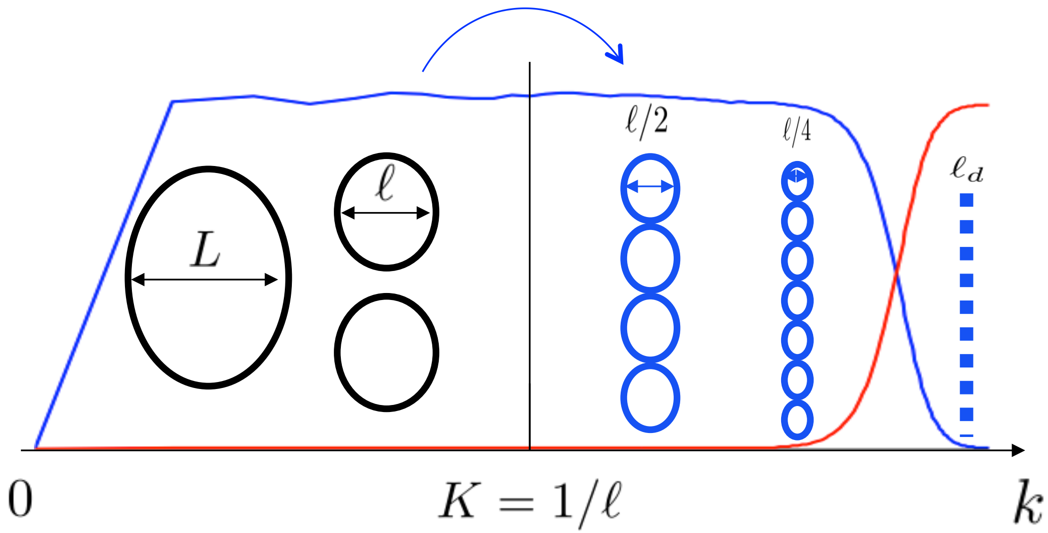
$$1/\overline{\rho}_\ell(\mathbf{x}) \leq \overline{(1/\rho)}_\ell(\mathbf{x}) \leq \|G_\ell\|_p\|\rho^{-1}\|_q = \|\rho^{-1}\|_q\left(\frac{L_{\rm dom}}{\ell}\right)^{d(1-\frac{1}{p})}\|G\|_p,$$

Compressible Turbulence

$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle \quad = -[\ldots] \quad + \text{ viscous dissipation}$$

<u>Conclusion I</u>

Viscosity localized to smallest scales when using Favre decomposition

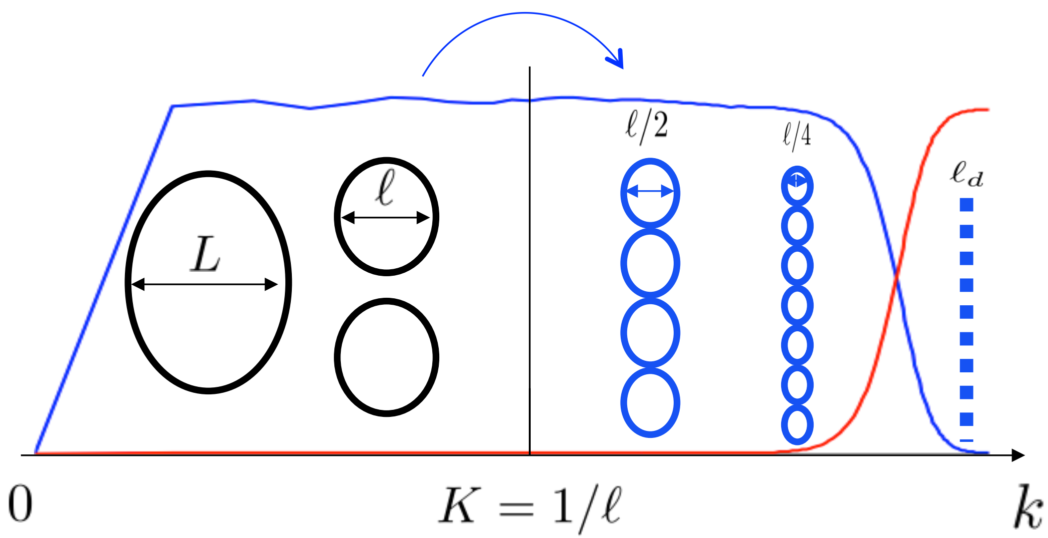


Compressible Turbulence

$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle = -[\ldots] \quad + \text{viscous dissipation}$$

Goal II

Scale-independent kinetic energy flux



Hydrodynamics

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

$$\partial_t (\rho \mathbf{u}) + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}\mathbf{u}) = -\boldsymbol{\nabla} P + \mu(\nabla^2 \mathbf{u} + \frac{1}{3} \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbf{u})$$

Energy budgets

$$\partial_t (\rho \frac{|\mathbf{u}|^2}{2}) + \boldsymbol{\nabla} \cdot [\ldots] = + P \boldsymbol{\nabla} \cdot \mathbf{u} -$$ viscous dissipation

$$\partial_t (\rho e) + \boldsymbol{\nabla} \cdot [\ldots] = - P \boldsymbol{\nabla} \cdot \mathbf{u} +$$ viscous dissipation

Hydrodynamics

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

$$\partial_t (\rho \mathbf{u}) + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}\mathbf{u}) = -\boldsymbol{\nabla} P + \mu(\nabla^2 \mathbf{u} + \frac{1}{3} \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbf{u})$$

Energy budgets

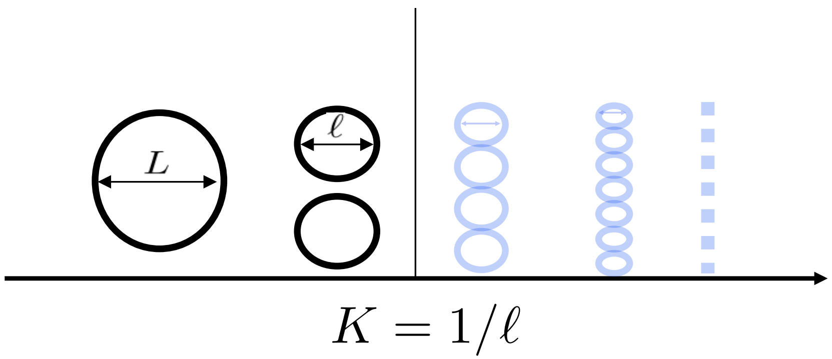
$$\partial_t (\rho \frac{|\mathbf{u}|^2}{2}) + \boldsymbol{\nabla} \cdot [\ldots] = +P \boldsymbol{\nabla} \cdot \mathbf{u} - \text{viscous dissipation}$$

$$\partial_t (\rho e) + \boldsymbol{\nabla} \cdot [\ldots] = -P \boldsymbol{\nabla} \cdot \mathbf{u} + \text{viscous dissipation}$$

pressure dilatation

Kinetic Energy Budget

$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle = -[\cdots] \quad + \langle\overline{P}_\ell \boldsymbol{\nabla}\cdot\overline{\mathbf{u}}_\ell\rangle$$ + viscous dissipation

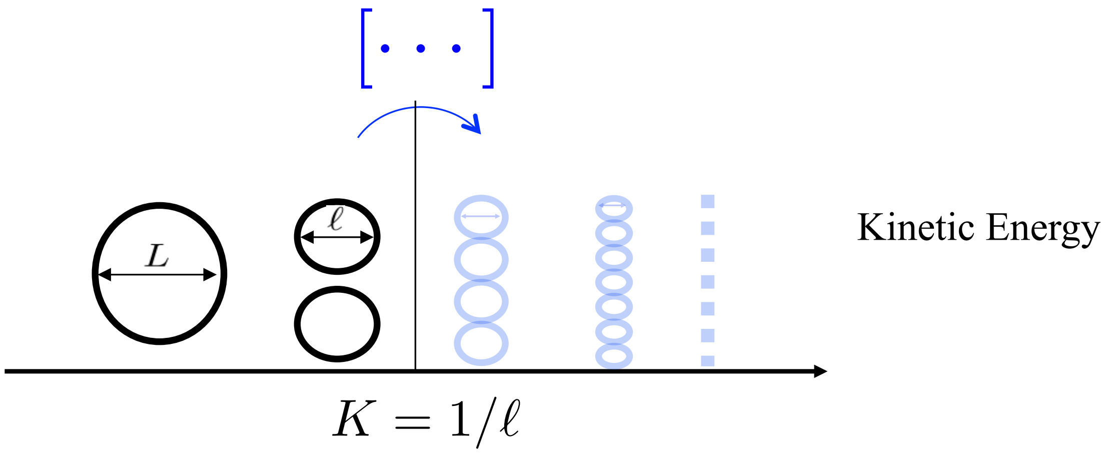


Kinetic Energy

Aluie H., *Phys. Rev. Lett.* (2011)

Kinetic Energy Budget

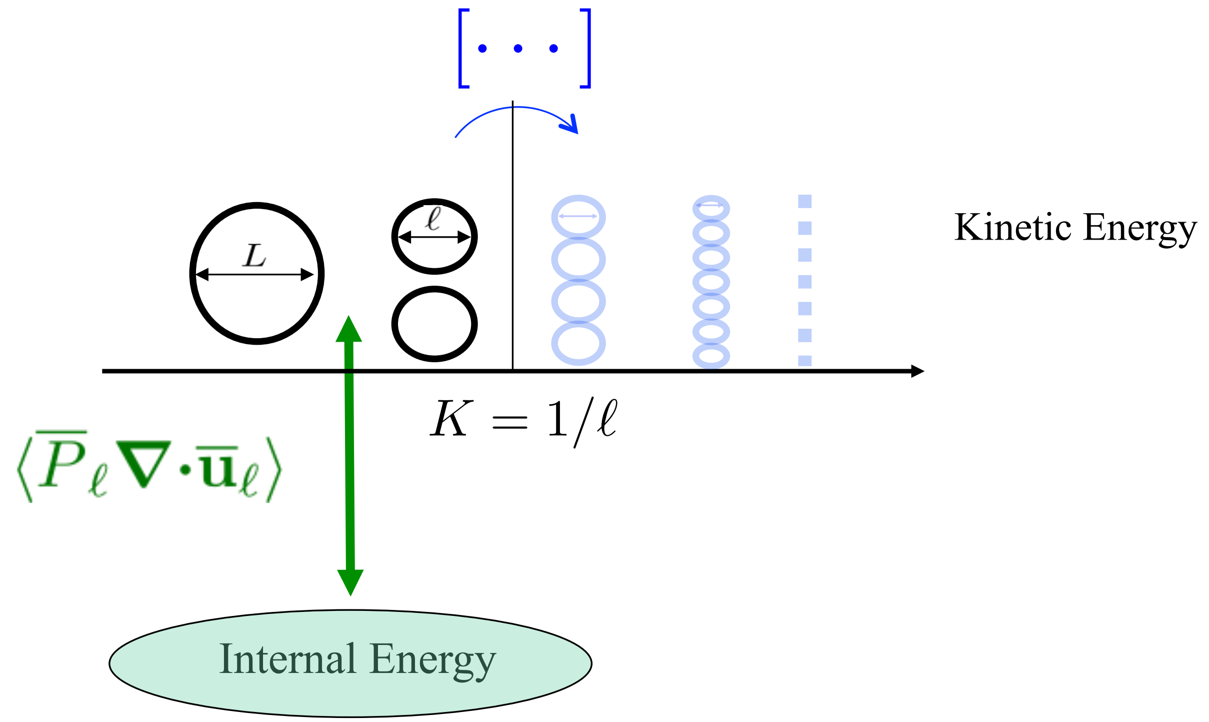
$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle = -[\cdots] \quad + \langle\overline{P}_\ell \nabla\cdot\overline{\mathbf{u}}_\ell\rangle \text{ + viscous dissipation}$$



Aluie H., *Phys. Rev. Lett.* (2011)

Kinetic Energy Budget

$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle = -[\cdots] \quad + \langle\overline{P}_\ell\boldsymbol{\nabla}\cdot\overline{\mathbf{u}}_\ell\rangle \text{ + viscous dissipation}$$



Aluie H., *Phys. Rev. Lett.* (2011)

Conversion

$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle = -[\cdots] \quad + \langle\overline{P}_\ell \nabla\cdot\overline{\mathbf{u}}_\ell\rangle \text{ + viscous dissipation}$$

Co-spectrum: $E^{PD}(k) = \sum_{k-0.5<|\mathbf{k}|<k+0.5} \widehat{P}(-\mathbf{k})\widehat{\nabla\cdot\mathbf{u}}(\mathbf{k})$

Aluie H., *Phys. Rev. Lett.* (2011)

Conversion

$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle = -[\cdots] \quad +\langle\overline{P}_\ell\boldsymbol{\nabla}\cdot\overline{\mathbf{u}}_\ell\rangle \text{ + viscous dissipation}$$

Sufficient Condition $|E^{PD}(k)| \leq (\text{const.})\ k^{-\beta} \qquad \beta > 1$

Co-spectrum: $E^{PD}(k) = \sum_{k-0.5<|\mathbf{k}|<k+0.5} \widehat{P}(-\mathbf{k})\widehat{\boldsymbol{\nabla}\cdot\mathbf{u}}(\mathbf{k})$

Aluie H., *Phys. Rev. Lett.* (2011)

Compressible Turbulence

$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle = -[\cdots] \quad + \langle\overline{P}_\ell \boldsymbol{\nabla}\cdot\overline{\mathbf{u}}_\ell\rangle$$ + viscous dissipation

---

Sufficient Condition $\quad |E^{PD}(k)| \leq (\text{const.})\ k^{-\beta} \qquad \beta > 1$

Co-spectrum: $\quad E^{PD}(k) = \sum_{k-0.5<|\mathbf{k}|<k+0.5} \widehat{P}(-\mathbf{k})\widehat{\boldsymbol{\nabla}\cdot\mathbf{u}}(\mathbf{k}) \qquad \langle\overline{P}_\ell \boldsymbol{\nabla}\cdot\overline{\mathbf{u}}_\ell\rangle = \sum_{0<k<K} E^{PD}(k)$

Aluie H., *Phys. Rev. Lett.* (2011)

Compressible Turbulence

$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle = -[\cdots] \quad + \langle\overline{P}_\ell\boldsymbol{\nabla}\cdot\overline{\mathbf{u}}_\ell\rangle \text{ + viscous dissipation}$$

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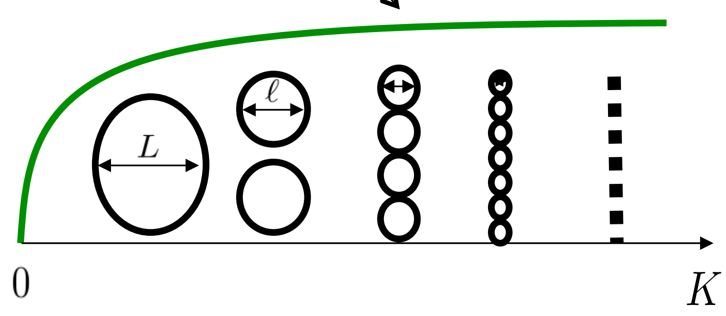


Aluie H., *Phys. Rev. Lett.* (2011)

Compressible Turbulence

$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle = -[\cdots] \quad +\langle\overline{P}_\ell\boldsymbol{\nabla}\cdot\overline{\mathbf{u}}_\ell\rangle \text{ + viscous dissipation}$$

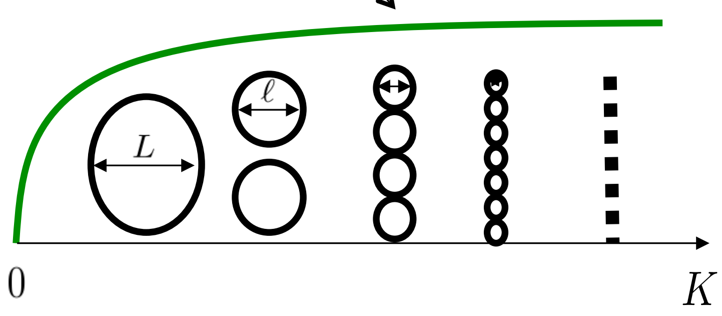
Sufficient Condition $\quad |E^{PD}(k)| \leq (\text{const.})\ k^{-\beta} \quad\quad \beta > 1$

Co-spectrum: $E^{PD}(k) = \sum_{k-0.5<|\mathbf{k}|<k+0.5} \widehat{P}(-\mathbf{k})\widehat{\boldsymbol{\nabla}\cdot\mathbf{u}}(\mathbf{k})$

$$\langle\overline{P}_\ell\boldsymbol{\nabla}\cdot\overline{\mathbf{u}}_\ell\rangle = \sum_{0<k<K} E^{PD}(k)$$

Convergence of series

$$\sum_{n=1}^{N}\frac{1}{n^\beta} = (\text{const.}) < \infty \quad \text{if} \quad \beta > 1$$



Aluie H., *Phys. Rev. Lett.* (2011)

Conversion

$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle = -[\cdots] \quad + \langle\overline{P}_\ell\boldsymbol{\nabla}\cdot\overline{\mathbf{u}}_\ell\rangle \text{ + viscous dissipation}$$

Sufficient Condition $\quad |E^{PD}(k)| \leq (\text{const.})\ k^{-\beta} \qquad \beta > 1$

Co-spectrum: $E^{PD}(k) = \sum_{k-0.5<|\mathbf{k}|<k+0.5} \widehat{P}(-\mathbf{k})\widehat{\boldsymbol{\nabla}\cdot\mathbf{u}}(\mathbf{k})$

1024^3 simulations

Run	Flow	Forcing	EOS	M_t
I	steady	solenoidal+compressive	isothermal	0.44
II	decaying	-	ideal gas	-
III	steady	solenoidal	isothermal	1.25
IV	decaying	-	ideal gas	-

Aluie H., Li S., Li H., *ApJ. Lett.* (2012)

Conversion

$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle = -[\cdots] \quad + \langle\overline{P}_\ell \boldsymbol{\nabla}\cdot\overline{\mathbf{u}}_\ell\rangle$$ + viscous dissipation

Sufficient Condition $\quad |E^{PD}(k)| \le (\text{const.})\ k^{-\beta} \quad \beta > 1$

Co-spectrum: $E^{PD}(k) = \sum_{k-0.5<|\mathbf{k}|<k+0.5} \widehat{P}(-\mathbf{k})\widehat{\boldsymbol{\nabla}\cdot\mathbf{u}}(\mathbf{k})$

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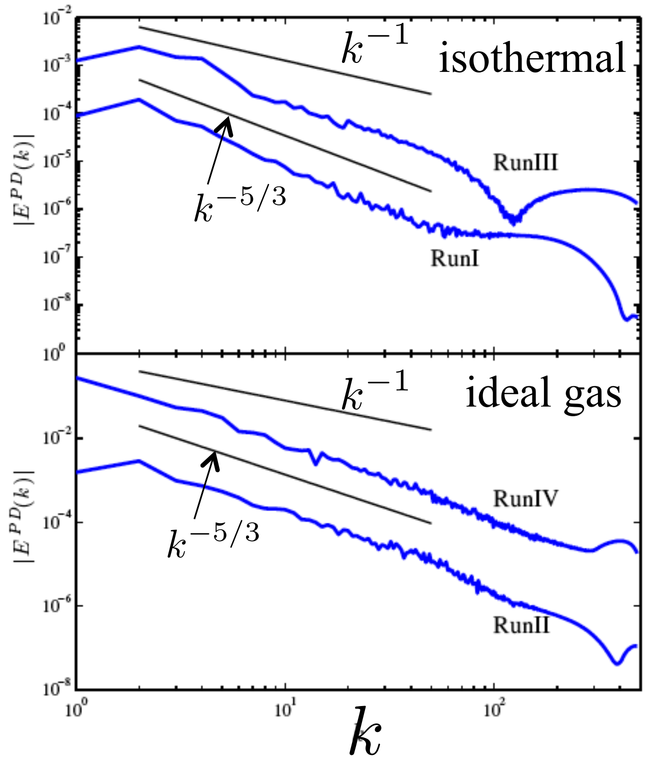
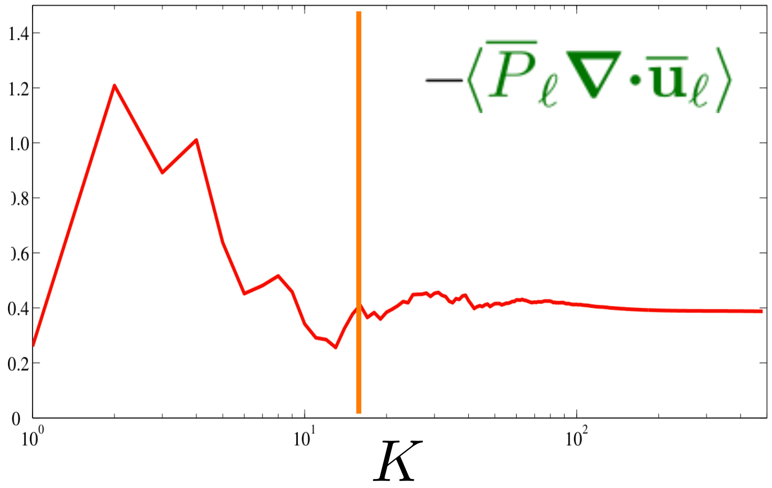
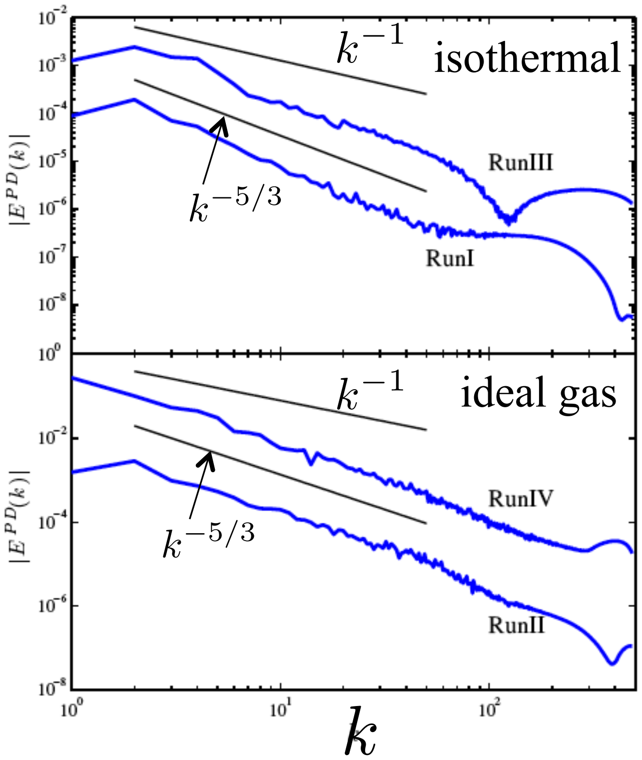


Aluie H., Li S., Li H., *ApJ. Lett.* (2012)

Conversion

$$\frac{1}{2}\frac{d}{dt}\langle\overline{\rho}_\ell|\widetilde{\mathbf{u}}_\ell|^2\rangle = -[\cdots] \quad + \langle\overline{P}_\ell\nabla\cdot\overline{\mathbf{u}}_\ell\rangle \text{ + viscous dissipation}$$

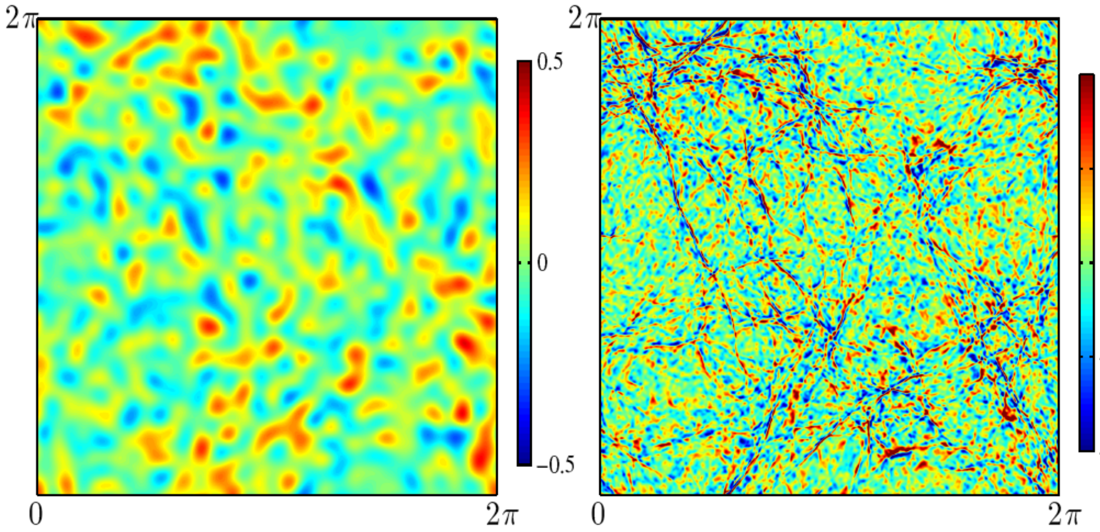




Cumulative function:

$$\langle\overline{P}_\ell\nabla\cdot\overline{\mathbf{u}}_\ell\rangle = \sum_{0<k<K} E^{PD}(k)$$

Aluie H., Li S., Li H., *ApJ. Lett.* (2012)

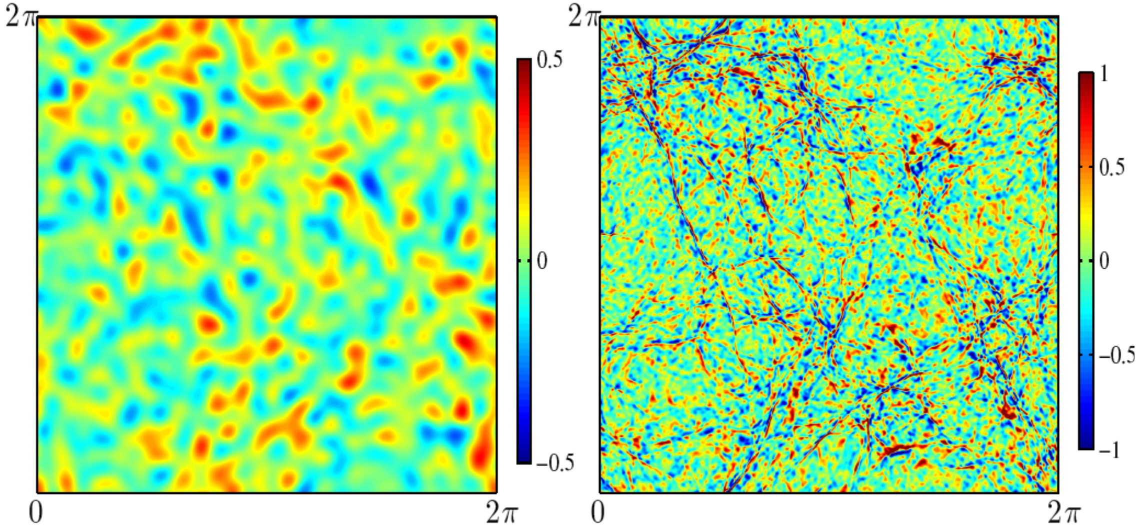


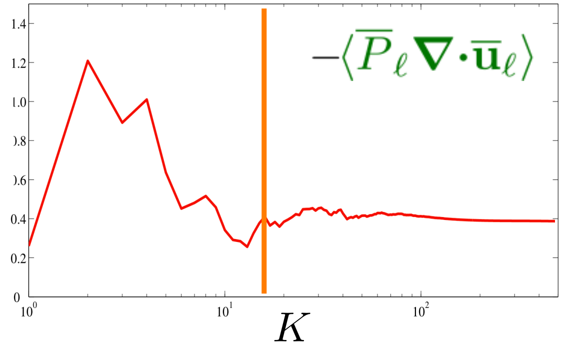
Large-scale conversion

$\overline{P}\nabla \cdot \overline{\mathbf{u}}(\mathbf{x})$

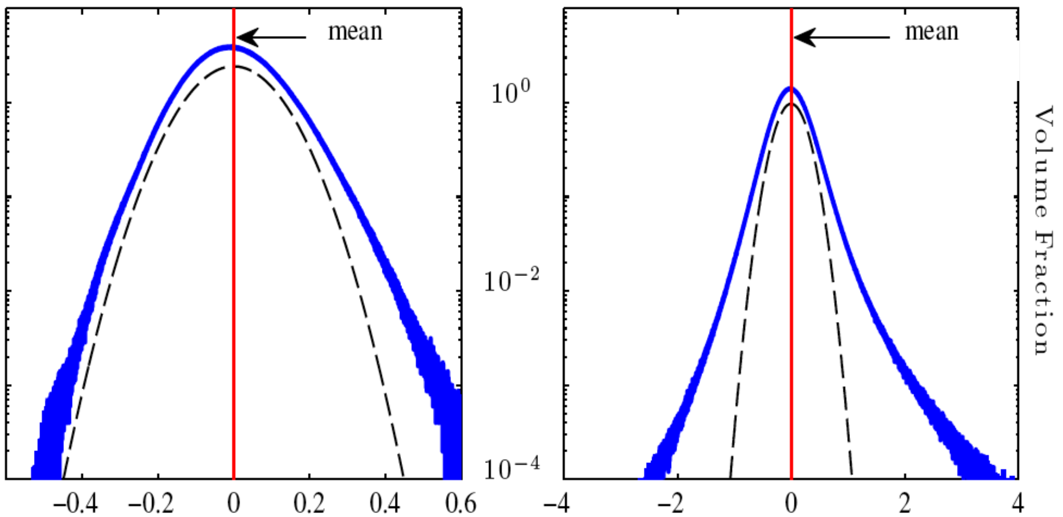
small-scale conversion

$(P\boldsymbol{\nabla}\!\cdot\!\mathbf{u} - \overline{P}\boldsymbol{\nabla}\!\cdot\!\overline{\mathbf{u}})(\mathbf{x})$





$= 0.4 \times 10^{-4}$ $= 0.2 \times 10^{-5}$



$\overline{P}\nabla \cdot \overline{\mathbf{u}}(\mathbf{x})$ $(P\boldsymbol{\nabla}\cdot\mathbf{u} - \overline{P}\boldsymbol{\nabla}\cdot\overline{\mathbf{u}})(\mathbf{x})$

Significant decorrelation

$$\langle P \boldsymbol{\nabla} \cdot \mathbf{u} \rangle$$

Conclusion

$$\partial_t\left(\rho\frac{|\mathbf{u}|^2}{2}\right) + \boldsymbol{\nabla}\cdot[\ldots] = +P\boldsymbol{\nabla}\cdot\mathbf{u} - \text{viscous dissipation}$$

$$\partial_t(\rho e) + \boldsymbol{\nabla}\cdot[\ldots] = -P\boldsymbol{\nabla}\cdot\mathbf{u} + \text{viscous dissipation}$$

1. Mean pressure-dilatation is a large-scale mechanism

2. Kinetic energy cascades conservatively despite not being an invariant

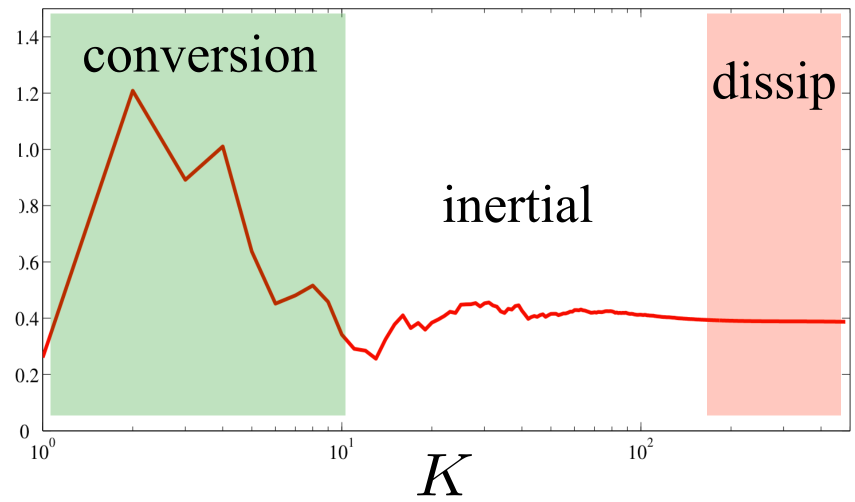
Aluie H., *Phys. Rev. Lett.* (2011)

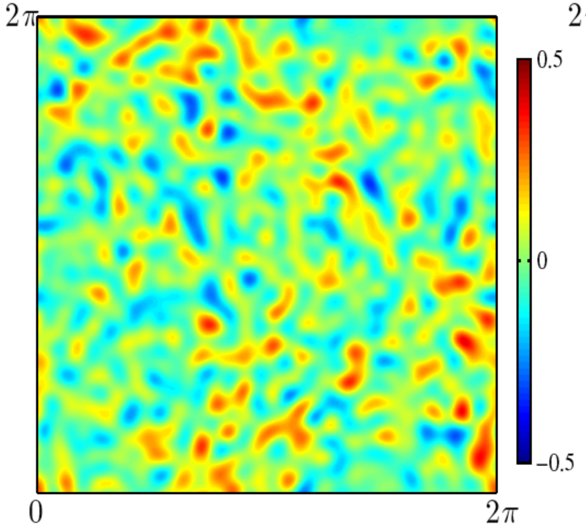
Aluie H., *Physica D* (2013)

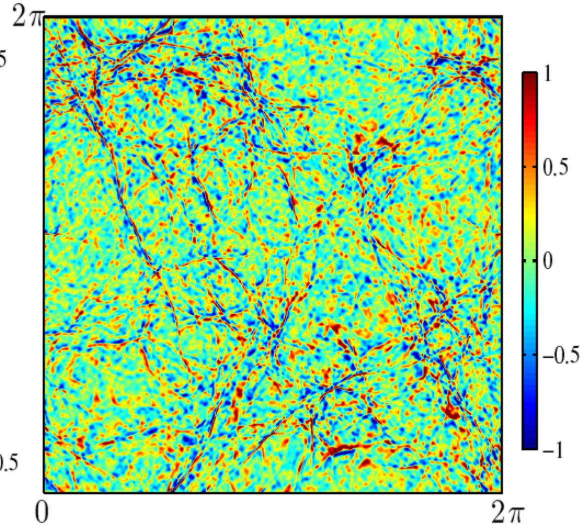
Aluie H., Li S., Li H., *ApJ. Lett.* (2012)

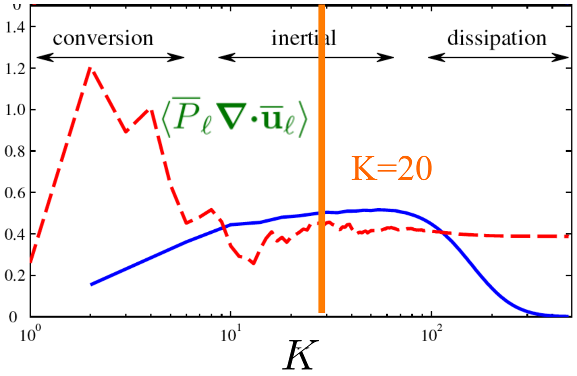
Chen S. *et al.*, *Phys. Rev. Lett.* (2013)

Kritsuk A. *et al.* JFM (2013)

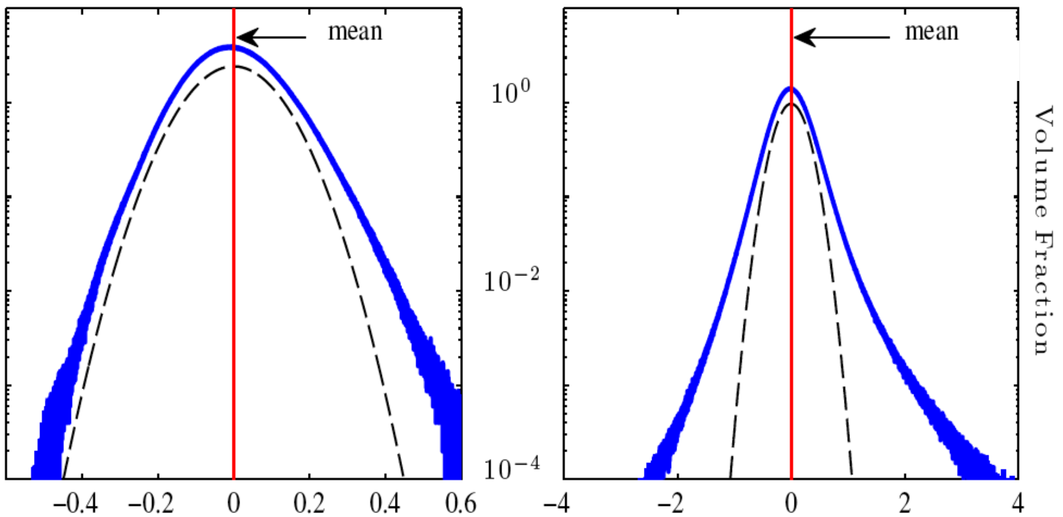








$= 0.4 \times 10^{-4}$ $= 0.2 \times 10^{-5}$



$\overline{P}\nabla \cdot \overline{\mathbf{u}}(\mathbf{x})$ $(P\boldsymbol{\nabla}\cdot\mathbf{u} - \overline{P}\boldsymbol{\nabla}\cdot\overline{\mathbf{u}})(\mathbf{x})$

Significant decorrelation

$$\langle P\boldsymbol{\nabla}\cdot\mathbf{u}\rangle$$

Conclusion

1. Favre based decomposition arises naturally from the sole requirement that viscous contributions to large-scale dynamics be negligible.

2. Mean pressure dilatation acts primarily on large-scales. Kinetic and internal energy budgets statistically decouple beyond a transitional “conversion” range.

3. Beyond the transitional “conversion” range, there is an inertial range over which kinetic energy cascades conservatively.

4. The cascade is scale-local despite the presence of shocks.

5. Scaling of density, pressure, and velocity spectra.